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A short proof of Grünbaum's conjecture about affine invariant points ☆

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ABSTRACT

Let us denote by $\mathcal{K}_n$ the hyperspace of all convex bodies of $\mathbb{R}^n$ equipped with the Hausdorff distance topology. An affine invariant point p is a continuous and $\mathrm{Aff}(n)$-equivariant map $p : \mathcal{K}_n \to \mathbb{R}^n$, where $\mathrm{Aff}(n)$ denotes the group of all nonsingular affine maps of $\mathbb{R}^n$. For every $K \in \mathcal{K}_n$, let $\mathfrak{P}_n(K) = \{p(K) \in \mathbb{R}^n \mid p \text{ is an affine invariant point}\}$ and $\mathfrak{F}_n(K) = \{x \in \mathbb{R}^n \mid gx = x \text{ for every } g \in \mathrm{Aff}(n) \text{ such that } gK = K\}$. In 1963, B. Grünbaum conjectured that $\mathfrak{P}_n(K) = \mathfrak{F}_n(K)$ [3]. After some partial results, the conjecture was recently proven in [6]. In this short note we give a rather different, simpler and shorter proof of this conjecture, based merely on the topology of the action of $\mathrm{Aff}(n)$ on $\mathcal{K}_n$.

1. Introduction

By a convex body $K \subset \mathbb{R}^n$ we mean a compact convex subset of $\mathbb{R}^n$ with a non-empty interior. We denote by $\mathcal{K}_n$ the set of all convex bodies of $\mathbb{R}^n$ equipped with the Hausdorff distance

$$d_H(A, B) = \max\left\{\sup_{b\in B} d(b, A), \qquad \sup_{a\in A} d(a, B)\right\},$$

where d is the standard Euclidean metric on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, $K \in \mathcal{K}_n$ and $\varepsilon > 0$ we will use the following notations:

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$$B(x,\varepsilon) = \{y \in \mathbb{R}^n \mid d(x,y) < \varepsilon\},$$
$$O(K,\varepsilon) = \{A \in \mathcal{K}_n \mid d_H(K,A) < \varepsilon\}.$$

Let us denote by $\mathrm{Aff}(n)$ the group of all nonsingular affine maps of $\mathbb{R}^n$. Namely, $g \in \mathrm{Aff}(n)$ if and only if there exist an invertible linear map $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ and a fixed point $v \in \mathbb{R}^n$ such that

$$gx = v + \sigma x, \quad \text{for every } x \in \mathbb{R}^n.$$

Observe that each element $g \in \mathrm{Aff}(n)$ satisfies that

$$g(tx + (1-t)y) = tgx + (1-t)gy, \quad \text{for every } x, y \in \mathbb{R}^n, \text{ and } t \in [0,1].$$

The natural action of $\mathrm{Aff}(n)$ on $\mathbb{R}^n$ induces a continuous action of $\mathrm{Aff}(n)$ on $\mathcal{K}_n$ by means of the formula

$$(g,K) \longmapsto gK, \quad gK = \{gx \mid x \in K\}.$$

We direct the reader to [1] for more details about this action.

In his famous paper [3], B. Grümbaum introduced the notion of an affine invariant point. Namely, an *affine invariant point* is a continuous map $p : \mathcal{K}_n \to \mathbb{R}^n$ such that

$$gp(K) = pg(K), \text{ for every } g \in \mathrm{Aff}(n), \text{ and } K \in \mathcal{K}_n.$$

The centroid, the center of John's ellipsoid or the center of Löwner's ellipsoid are examples of affine invariant points. If $p : \mathcal{K}_n \to \mathbb{R}^n$ is an affine invariant point and satisfies

$$p(A) \in \mathrm{int}(A) \quad \text{for every } A \in \mathcal{K}_n$$

then we call p a *proper affine invariant point*. Using standard notation, let us denote by $\mathfrak{P}_n$ the set of all affine invariant points of $\mathcal{K}_n$. For every $K \in \mathcal{K}_n$, let $\mathfrak{P}_n(K)$ be the set of all $x \in \mathbb{R}^n$ such that $x = p(K)$ for some $p \in \mathfrak{P}_n$ and

$$\mathfrak{F}_n(K) = \{x \in \mathbb{R}^n \mid gx = x \text{ for every } g \in \mathrm{Aff}(n) \text{ such that } gK = K\}.$$

Observe that if $x \in \mathfrak{P}_n(K)$, then there exists $p \in \mathfrak{P}_n$ with $p(K) = x$. Therefore, if $g \in \mathrm{Aff}(n)$ satisfies $gK = K$ then we have that

$$x = p(K) = p(gK) = gp(K) = gx$$

and thus $x \in \mathfrak{F}_n(K)$. This implies that $\mathfrak{P}_n(K) \subset \mathfrak{F}_n(K)$. In [3] Grünbaum conjectured that $\mathfrak{P}_n(K) = \mathfrak{F}_n(K)$, and in the following 50 years several partial results were obtained (see, e.g., [4,5]). Recently a full proof of this conjecture was finally given by O. Mordhorst in [6].

The aim of this work is to provide an alternative proof of Grümbaum's conjecture which is simpler, shorter and different from the one given in [6]. Our proof is based merely on some well-known topological results. What is interesting about this way of approaching Grümbaum's conjecture (which was open for more than 50 years) is that once we translate the problem to the language of topological transformation groups theory, the proof is quite natural and simple.