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Some results about inverse limits with set-valued bonding functions ☆

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ABSTRACT

In the paper, we give new results about dimension of inverse limits with upper semicontinuous bonding functions. We also construct an upper semicontinuous function $f : [0,1] \to 2^{[0,1]}$ with the following properties: (1) the graph of f is an arc, (2) the graph of f is surjective, (3) the inverse limit $\varprojlim\{[0,1], f\}_{n=1}^{\infty}$ contains a simple closed curve, and (4) the dimension of $\varprojlim\{[0,1], f\}_{n=1}^{\infty}$ is 1. This answers a question by W.T. Ingram.

1. Introduction

In 2004, inverse limits of inverse sequences of compact metric spaces with upper semicontinuous bonding functions were introduced by W.T. Ingram and W.S. Mahavier [5,6]. They are a generalization of inverse limits of inverse sequences of compact metric spaces with continuous bonding functions.

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I. Banič and J. Kennedy drew attention to inverse limits of inverse sequences of unit intervals $[0,1]$ with a single upper semicontinuous bonding function whose graph is an arc [1]. They are a natural generalization of inverse limits of inverse sequences of unit intervals with a single continuous bonding function, since the graph of such a function is also an arc. This is a largely unexplored field of study in the theory of inverse limits of inverse sequences of compact metric spaces with upper semicontinuous bonding functions. In [3], W.T. Ingram posed a number of open problems concerning such inverse limits. One of them states:

Problem 1.1. ([3, Problem 4.15]) Suppose $f : [0,1] \to 2^{[0,1]}$ is an upper semicontinuous set-valued function whose graph is an arc. If the inverse limit $\varprojlim\{[0,1], f\}_{n=1}^{\infty}$ is one-dimensional, can it contain a simple closed curve?

One of the purposes of this paper is to give the answer to that question in the positive.

Another purpose of this paper is to provide easy tools for determining dimension of inverse limits of inverse sequences of compact metric spaces with upper semicontinuous bonding functions. The research of the dimension of such inverse limits has been very intensive since their introduction; for more information see [4]. In present paper, we give new results about the dimension of such inverse limits – the results provide easy tools for determining dimension as well as for finding upper bounds for the dimension of such inverse limits. We demonstrate this when examining the inverse limit in the last section.

We proceed as follows. In Section 2, basic definitions, notation and some known results (that will be used later) are given. In Section 3, we give results about the dimension of inverse limits with upper semicontinuous bonding functions. At the end, in Section 4, we construct a function $f : [0,1] \to 2^{[0,1]}$ with the following properties:

1. the graph of f is an arc,
2. the graph of f is surjective,
3. the inverse limit $\varprojlim\{[0,1], f\}_{n=1}^{\infty}$ contains a simple closed curve, and
4. $\dim(\varprojlim\{[0,1], f\}_{n=1}^{\infty}) = 1$.

2. Definitions and notation

If (X,d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X.

A function $f : X \to 2^Y$ is *upper semicontinuous at the point* $x \in X$ provided that if V is any open set in Y containing $f(x)$ then there is an open set U in X containing x such that $f(t) \subseteq V$ for any $t \in U$; f is called *upper semicontinuous* (abbreviated *u.s.c.*) if it is upper semicontinuous at each point of X.

The *graph* $\Gamma(f)$ *of a function* $f : X \to 2^Y$ is the set of all points $(x,y) \in X \times Y$ such that $y \in f(x)$. We say that the graph of a function $f : X \to 2^Y$ is *surjective* if for each $y \in Y$ there is a point $x \in X$ such that $y \in f(x)$.

Ingram and Mahavier gave the following characterization of u.s.c. functions [4, p. 3]:

Theorem 2.1. *Let X and Y be compact metric spaces and $f : X \to 2^Y$ a function. Then f is u.s.c. if and only if its graph $\Gamma(f)$ is closed in $X \times Y$.*

In this paper we deal with *inverse sequences* $\{X_n, f_n\}_{n=1}^{\infty}$, where X_n are compact metric spaces and $f_n : X_{n+1} \to 2^{X_n}$ u.s.c. functions.

The *inverse limit* of an inverse sequence $\{X_n, f_n\}_{n=1}^{\infty}$ is defined to be the subspace of the product space $\prod_{n=1}^{\infty} X_n$ consisting of all $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{n=1}^{\infty} X_n$, such that $x_n \in f_n(x_{n+1})$ for each n. The inverse limit is denoted by $\varprojlim\{X_n, f_n\}_{n=1}^{\infty}$.

Let $\{X_n, f_n\}_{n=1}^{\infty}$ be an inverse sequence of compact metric spaces with u.s.c. bonding functions. Let $G_n = \Gamma(f_n)$ for each positive integer n. Then