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Two-stage spaces and the torus rank conjecture

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ABSTRACT

In this note, we give some new families of two-stage spaces for which the torus rank conjecture is affirmed.

1. Introduction

The torus rank conjecture is one of the long-standing problems in algebraic topology. The conjecture says that if a nilpotent finite dimensional CW-complex admits an almost free T^n action, then

$$\dim H(X;\mathbb{Q}) \geq 2^n.$$

We shall henceforth assume that all spaces are 1-connected, finite cell complexes. Now, the conjecture is proved for $n \leq 3$ [2]. There are many families of spaces for which the conjecture holds. For instance it is proved that the conjecture holds for homogeneous spaces of a compact Lie group, and spaces satisfying the hard Lefschetz properties [3].

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There is an algebraic version of this conjecture [1], if M is an m-dimensional compact manifold, then there is an almost free T^n action on M if there is a relative Sullivan algebra of the form

$$(\Lambda(x_1, x_2, \cdots, x_n), 0) \to (\Lambda(x_1, x_2, \cdots, x_n) \otimes \Lambda V, D) \to (\Lambda V, d),$$

where $|x_i| = 2$ for $i = 1, 2, \cdots, n$, $(\Lambda V, d)$ is a minimal model of M and the cohomology groups $H^*(\Lambda(x_1, x_2, \cdots, x_n) \otimes \Lambda V, D)$ are finite dimensional. Moreover, if the above condition holds, then T^n acts freely on a finite CW-complex X which has the same homotopy type as M, and if $m - n \neq 0 \bmod 4$, we can choose X to be a compact manifold. In this case, the relative Sullivan algebra is a model of the associated Borel fibration of the torus action.

A Sullivan algebra is said to be a two-stage algebra if $(\Lambda V, d) \cong (\Lambda(U \oplus W), d)$ such that $dU = 0$ and $dW \subset \Lambda U$. We call the isomorphism $V \cong U \oplus W$ a two-stage decomposition of $(\Lambda V, d)$. If the minimal model of a space X is a two-stage Sullivan algebra, we say that X is a two-stage space.

A Sullivan algebra $(\Lambda W, d)$ with W and $H(\Lambda W, d)$ both finite-dimensional are called elliptic, and a space X with an elliptic minimal model is called an elliptic space. Topologically, this means that both $\pi(X) \otimes \mathbb{Q}$ and $H(X; \mathbb{Q})$ are finite dimensional.

In this paper, we will show the following theorem.

Theorem 1.1. *Let $(\Lambda V, d)$ be a two-stage algebra such that V is a finite dimensional graded vector space. If $k_V \leq 3$, the torus rank conjecture holds for this algebra, where k_V is defined in Section 2.*

2. Basics of rational homotopy theory

At first, we recall some basics of rational homotopy theory, all of them can be found in [4]. We simply remark a few facts.

We recall that when M is path connected, the Sullivan model of M is a quasi-isomorphism:

$$m : (\Lambda V_M, d) \to A_{PL}(M),$$

where $(\Lambda V_M, d)$ is a Sullivan algebra.

A minimal model of a map $f : X \to Y$ is a quasi-isomorphism:

$$\varphi : (\Lambda V_Y \otimes \Lambda W, D) \to (\Lambda V_X, d),$$

where $(\Lambda V_Y, d)$ and $(\Lambda V_X, d)$ are the models of Y and X, respectively, and also $Dw \in (\Lambda^+ V_Y \otimes \Lambda W) \oplus \Lambda^+ W$ for $w \in W$.

Given a nilpotent fibration $F \hookrightarrow X \to Y$ where the involved spaces are all of finite type, then there is a K-S extension model of the fibration given by

$$(\Lambda V_Y, d) \hookrightarrow (\Lambda V_Y \otimes \Lambda V_F, D) \to (\Lambda V_F, d),$$

where $(\Lambda V_Y \otimes \Lambda V_F, D)$ is a relative minimal model of the projection.

A pure Sullivan algebra is a special two-stage algebra $(\Lambda V, d)$, which has a two-stage decomposition $V = U \oplus W$ such that $U = V^{\text{even}}$.

Let $(\Lambda V, d)$ be a two-stage algebra such that $\dim V < \infty$ and $\dim H(\Lambda V, d) < \infty$. Let $V = U \oplus W$ be the two-stage decomposition of this algebra. We have that W is concentrated in odd degrees. Let $d = d_\tau + \theta$, where $d_\tau v \in \Lambda V^{\text{odd}}$ and $\theta v \in \Lambda^+ V^{\text{even}} \cdot \Lambda V$ for $v \in V$. Then $(\Lambda V, d_\tau)$ is also a two-stage algebra. We have a new two-stage decomposition $V = U' \oplus W'$ for the new algebra.