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A comparison of large scale dimension of a metric space to the dimension of its boundary ☆

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ABSTRACT

Buyalo and Lebedeva have shown that the asymptotic dimension of a hyperbolic group is equal to the dimension of the group boundary plus one. Among the work presented here is a partial extension of that result to all groups admitting $\mathcal{Z}$-structures; in particular, we show that $\text{asdim}G \geq \dim Z + 1$ where Z is the $\mathcal{Z}$-boundary.

1. Introduction

The primary goal of this paper is to establish a connection between the asymptotic dimension of a group admitting a $\mathcal{Z}$-structure and the covering dimension of the group's boundary.

For hyperbolic G, the relationship is strong; Buyalo and Lebedeva [5] have shown that $\text{asdim}G = \dim\partial G + 1$. In [6], a partial extension to CAT(0) groups was attempted. Specifically, it was claimed that $\text{asdim}G \geq \dim\partial G + 1$, where ∂G is any CAT(0) boundary of G. However, in MathSciNet review MR3058238, X. Xie pointed out a critical error in the proof. Here we recover the same inequality as a special case of a more general theorem.

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Theorem 1. *Suppose a group G admits a $\mathcal{Z}$-structure, $(\overline{X}, Z)$. Then* $\dim Z + 1 \leq \operatorname{asdim} G$.

By a $\mathcal{Z}$-structure on G, we are referring to the axiomatized approach to group boundaries laid out in [2] and expanded upon in [8]. Groups known to admit $\mathcal{Z}$-structures include: hyperbolic groups (with X being a Rips complex and $Z = \partial G$) [3]; CAT(0) groups (with X being the CAT(0) space and Z its visual boundary) [2]; systolic groups [15], Baumslag–Solitar groups [10]; as well as various combinations of these classes, as described in [17,7,13]. Definitions of $\mathcal{Z}$-structure and other key terms used here will be provided in the next section. Theorem 1 will be obtained from a more general observation about metric spaces.

Theorem 2. *Suppose a proper metric space (X, d) admits a controlled $\mathcal{Z}$-compactification $\overline{X} = X \cup Z$. Then* $\dim Z + 1 \leq \dim_{\mathrm{mc}} X$.

Here, $\dim_{\mathrm{mc}}$ stands for Gromov's *macroscopic dimension*, a type of large scale dimension for metric spaces that is less restrictive than asymptotic dimension in that, for any (X, d), $\dim_{\mathrm{mc}} X \leq \operatorname{asdim} X$. To complete the proof of Theorem 1 it will then suffice to show that, for a $\mathcal{Z}$-structure $(\overline{X}, Z)$ on a group G, $\overline{X}$ is a controlled $\mathcal{Z}$-compactification and $\operatorname{asdim} X = \operatorname{asdim} G$.

Theorem 2 is inspired by the main argument in [11] together with the point of view presented in [14].

2. Background and definitions

We begin by providing a few definitions and results for the different dimension theories and then we discuss controlled $\mathcal{Z}$-compactifications and $\mathcal{Z}$-structures.

Given a cover $\mathcal{U}$ of a metric space X, $\operatorname{mesh}\mathcal{U} = \sup\{\operatorname{diam}(U) | U \in \mathcal{U}\}$. The cover is **uniformly bounded** if there exists some $D > 0$ such that $\operatorname{mesh}\mathcal{U} \leq D$. The **order** of $\mathcal{U}$ is the smallest integer n for which each element $x \in X$ is contained in at most n elements of $\mathcal{U}$.

Definition 3. The **covering dimension** of a space X is the minimal integer n such that every open cover of X has an open refinement of order at most $n + 1$.

There are various ways to show that a space has finite covering dimension. When working with compact metric spaces, we prefer the following.

Lemma 4. *For a compact metric space X,* $\dim X \leq n$ *if and only if, for every $\epsilon > 0$, there is an open cover $\mathcal{U}$ of X with* $\operatorname{mesh}\mathcal{U} < \epsilon$ *and* $\operatorname{order}\mathcal{U} \leq n + 1$.

Covering dimension can be thought of as a small-scale property. Gromov introduced asymptotic dimension as a large scale analog of covering dimension [9].

Definition 5. The **asymptotic dimension** of a metric space X is the minimal integer n such that for every uniformly bounded open cover $\mathcal{V}$ of X, there is a uniformly bounded open cover $\mathcal{U}$ of X with $\operatorname{order}\mathcal{U} \leq n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. In this case, we write $\operatorname{asdim} X = n$.

We note here that the covers need not be open in the definition of asymptotic dimension. We chose to follow conventional definitions here. For a nice survey of asymptotic dimension, see [1]. Although Theorem 1 is stated for asymptotic dimension, we will prove a stronger result using a weaker notion of large scale dimension known as *(Gromov) macroscopic dimension*.

Definition 6. The **Gromov macroscopic dimension** of a metric space X is the minimal integer n such that there exists a uniformly bounded open cover of X with order at most $n + 1$. In this case, we write $\dim_{\mathrm{mc}} X = n$.