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On a problem of Mauldin and Ulam about continuous images

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ABSTRACT

In this paper we give a solution to a question of Mauldin and Ulam about transformations preserving continuous images.

1. Introduction

Let E and F be two topological (e.g., metric) spaces. Let T be a transformation of E into F such that if $A, B \subset E$ and B is a continuous image of A, then $T(B)$ is a continuous image of $T(A)$ (we will say that T is a CI-map). In ([4], IV.3.) Mauldin and Ulam ask if such a map T must be continuous.

The aim of this paper is to solve this problem in a fairly complete manner. In particular we show precisely when a CI-map defined on a sequential space is continuous.

The reader is referred to [2] for notations and terminology not explicitly given. All spaces considered here will be Hausdorff.

2. The results

A space E is called: (i) sequential if for every non-closed subset A of E there exists a sequence $(x_n)_n$ of points of A converging to a point of the set $\overline{A} \setminus A$; (ii) almost sequential if for every non-isolated point there

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is a non-trivial sequence converging to it; (iii) contrasequential provided it has no non-trivial convergent sequences (see [1]). Let us observe that every metric space is sequential. Moreover a map T between two topological spaces E and F is called sequentially continuous if for every sequence $(x_n)_n$ in E converging to a point $x \in X$, the image $(T(x_n))_n$ converges to the point $T(x)$. Every continuous map is sequentially continuous and every sequentially continuous map whose domain is sequential is continuous.

Our first result will show that every CI-map satisfying a rather mild condition on the image is sequentially continuous.

Theorem 1. *Let $T : E \to F$ be a CI-map where* $\mathrm{Im}(T)$ *is not contrasequential. Then T is sequentially continuous.*

Proof. First observe that T cannot be constant (otherwise $\mathrm{Im}(T)$ would be contrasequential). We may assume, without loss of generality, that T is onto. First let us show that T is injective. In fact let us suppose that there are two distinct points x and y of E such that $T(x) = T(y) = a$. Take $b \in \mathrm{Im}(T)$ and let $z \in E$ such that $T(z) = b \neq a$. Then $A = \{x, y\}$ and $B = \{x, z\}$ are homeomorphic (in particular B is a continuous image of A), while $T(B) = \{a, b\}$ is not a continuous image of $T(A) = \{a\}$. A contradiction. □

Claim. *There exists a sequence $(p_n)_n$ in E, with distinct terms, converging to a point p such that the sequence $(T(p_n))_n$ converges to $T(p)$.*

Proof of Claim. Let $(q_n)_n$ be a non-trivial sequence in F converging to a point q. We may assume that $q_n \neq q_m$ whenever $n \neq m$. Let p_n and x be such that $T(p_n) = q_n$ for every n and $T(p) = q$. Clearly $(p_n)_n$ is a sequence with distinct terms. We claim that $(p_n)_n$ converges to p. If not, we may assume, without loss of generality, that the set $A = \{p_n : n \in \mathbb{N}\} \cup \{p\}$ is discrete. In fact A contains an infinite subset S such that $p \notin \overline{S}$, so we may replace A with $D \cup \{p\}$ where D is an infinite discrete subset of S (recall that every infinite Hausdorff space contains an infinite discrete subset). Now set $B = \{q_n : n \in \mathbb{N}\} \cup \{q\}$, $C = A \setminus \{p\}$ and $D = B \setminus \{q\}$. Since A and C are homeomorphic and T is a CI-map, it follows that $T(C) = T(A \setminus \{p\}) = T(A) \setminus \{T(p)\} = B \setminus \{q\} = D$ is a continuous image of $T(A) = B$. Since B is compact and D is not, we reach a contradiction. Therefore the sequence $(p_n)_n$ converges to p. □

Now we are ready to show that T is sequentially continuous. So let us take a sequence $(x_n)_n$ in E converging to a point $x \in X$ and let us prove that the image $(T(x_n))_n$ converges to the point $T(x)$. Suppose not and let us set $A = \{p_n : n \in \mathbb{N}\} \cup \{p\}$ and $B = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. There is no loss of generality in assuming that $T(x)$ be isolated in $T(B)$. Since A and B are homeomorphic, it follows that $T(B)$ is a continuous image of the compact space $T(A)$. So $T(B)$ is compact too. Since $A \setminus \{p\}$ and $B \setminus \{x\}$ are homeomorphic (they are discrete and countably infinite) and T is a CI-map, it follows that the infinite discrete space $T(A \setminus \{p\}) = T(A) \setminus \{T(p)\}$ is a continuous image of the compact space $T(B \setminus \{x\}) = T(B) \setminus \{T(x)\}$. A contradiction. Therefore $(T(x_n))_n$ converges to the point $T(x)$, and T is sequentially continuous.

From Theorem 1 we readily obtain the following

Corollary 1. *Let $T : E \to F$ be a CI-map where E is sequential and* $\mathrm{Im}(T)$ *is not contrasequential. Then T is continuous.*

Observe that the first part of the proof of Theorem 1 shows that every non-constant CI-map must be injective, so we have

Corollary 2. *Let $T : E \to F$ be a non-constant CI-map where E is a non-discrete sequential space. T is continuous if and only if* $\mathrm{Im}(T)$ *is not contrasequential.*