Topology and its Applications 199 (2016) 95–101

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

A note on duals of topologies ☆

Xiaoquan Xu *

Department of Mathematics, Nanchang Normal University, Nanchang 330032, China

ARTICLE INFO

Article history:
Received 13 September 2014
Received in revised form 8 October 2015
Accepted 7 December 2015
Available online 23 December 2015

MSC:
06B35
06F30
54B99
54D30

Keywords:
Dual of topology
Lower topology
Scott topology
Continuous domain
Quasicontinuous domain

ABSTRACT

In this note it is proved that for a quasicontinuous lattice L, the lower topology $\omega(L)$ and the Scott topology $\sigma(L)$ are duals for each other; and if L is a complete lattice such that $\sigma(L)$ is continuous but not hypercontinuous (equivalently, L is not quasicontinuous), then $\omega(L)$ is not the dual of $\sigma(L)$ and hence they are not duals for each other.

1. Introduction

For a topological space (X, τ), let τ^d be the (*de Groot*) *dual* of τ and let $\tau^{dd} = (\tau^d)^d$ (see [6–8]). In [7] Kovár has given some important conditions under which $\tau = \tau^{dd}$. Using one of these conditions he [8] has presented an example which may illustrate the important and interesting relationship between the Scott topology and compactness in terms of dual.

In this note we investigate the duality of the lower topology and Scott topology from the point of view of domain theory. It is proved that for a quasicontinuous lattice L, the lower topology $\omega(L)$ and the Scott topology $\sigma(L)$ are duals for each other. On the other hand, it is proved that if L is a complete lattice such

☆ This research is supported by the National Natural Science Foundation of China (No. 11161023), the Fund for the Author of National Excellent Doctoral Dissertation of China (No. 2007B14), the Ganpo 555 project for leading talents of Jiangxi Province and the NFS of Jiangxi Province (No. 20114BAB201008).

* Tel.: +86 791 88120360; fax: +86 791 83812190.
E-mail address: xiqxu2002@163.com.

http://dx.doi.org/10.1016/j.topol.2015.12.006

that $\sigma(L)$ is continuous but not hypercontinuous (equivalently, L is not quasicontinuous), then $\omega(L)$ is not the dual of $\sigma(L)$ and hence they are not duals for each other.

2. Preliminaries

In this section we recall some basic definitions and notations used in this note; more details can be found in [2,6–8]. For a poset P and $A \subseteq P$, let $\uparrow A = \{x \in P : a \leq x$ for some $a \in A\}$ (dually $\downarrow A = \{x \in P : x \leq a$ for some $a \in A\}$) and $\mathbf{up}(P) = \{A \subseteq P : A = \uparrow A\}$. The family of all finite sets in P is denoted by $P^{(<\omega)}$. Let $\mathbf{Fin}P = \{\uparrow F : F \in P^{(<\omega)} \backslash \{\emptyset\}\}$. $\mathbf{Fin}P$ is always endowed with the order of inverse inclusion of sets when it is considered as a poset. P is said to be a *directed complete poset*, a *dcpo* for short, if every directed subset of P has the least upper bound in P. The topology generated by the collection of sets $P \backslash \uparrow x$ (as subbasic open subsets) is called the *lower topology* on P and denoted by $\omega(P)$; dually define the *upper topology* on P and denote it by $\upsilon(P)$. The joint topology $\theta(P) = \omega(P) \vee \omega(P)$ is called the *interval topology*. The topology $\sigma(P) = \{U \subseteq P : U = \uparrow U$ and $U \cap D \neq \emptyset$ for each directed set D with $\vee D \in U\}$ is called the *Scott topology*. The joint topology $\omega(P) \vee \sigma(P)$ is called the *Lawson topology* and is denoted by $\lambda(P)$. Let P^{op} denote the dual of P.

Given a topological space (X, τ), we can define a preorder $\leq_\tau$, called the *preorder of specialization*, by $x \leq_\tau y$ if and only if $x \in cl_\tau\{y\}$. Clearly, each open set is an upper set and each closed set is a lower set with respect to the preorder $\leq_\tau$. It is easy to see that $\leq_\tau$ is a partial order if and only if (X, τ) is a T_0 space. For any set $A \subseteq X$ we denote $\uparrow_\tau A = \{x \in X : a \leq_\tau x$ for some $a \in A\}$ and $\downarrow_\tau A = \{x \in X : x \leq_\tau a$ for some $a \in A\}$. For $A = \{x\}$, $\uparrow_\tau A$ and $\downarrow_\tau A$ are shortly denoted by $\uparrow_\tau x$ and $\downarrow_\tau x$ respectively. A set is said to be *saturated* in (X, τ) if it is the intersection of open sets, or equivalently if A is an upper set (that is, $A = \uparrow_\tau A$). We define the (*de Groot*) *dual* τ^d of the original topology τ by taking as a subbasis for the closed sets all compact saturated sets in (X, τ) (see [3,5–8]). The topology $\tau^{dd} (= (\tau^d)^d)$ is called the *double dual* of τ. If $\tau = \tau^{dd}$, then τ is said to be *doubly self-dual*. When τ is considered as a complete lattice, it is always endowed with the order of set inclusion. (X, τ) is said to be *sober* if it is T_0 and every irreducible closed set is a closure of a (unique) singleton.

In the following, let Φ always denote a certain closed subbasis of a topological space (X, τ) and Φ^d be the collection of all compact saturated sets in (X, τ) and hence a closed subbasis of (X, τ^d). The collection of all compact saturated sets in (X, τ^d) (hence a closed subbasis of (X, τ^{dd})) is denoted by Φ^{dd}. A family $\Psi \subseteq 2^X$ is said to have the *finite intersection property*, or briefly, that Ψ has *FIP*, if $P_1 \cap P_2 \cap \ldots \cap P_k \neq \emptyset$ for every finite family $\{P_1, P_2, \ldots, P_k\} \subseteq \Psi$. It is well known from the Alexander's subbasis lemma that $S \subseteq X$ is compact in (X, τ) if and only if $S \cap (\bigcap \Re) \neq \emptyset$ for every family $\Re \subseteq \Phi$ such that $\{S\} \cup \Re$ has FIP (see [2]). The family Ψ is said to be *up-compact* if every $A \in \Psi$ is compact with respect to the family $\{\uparrow x : x \in X\}$ (that is, every $A \in \Psi$ is compact in $(X, \omega(X, \leq_\tau))$). We say that the family Φ is Ψ-*down-conservative* if for every $A \in \Phi$ and $B \in \Psi$ it follows $\downarrow(A \cap B) \in \Phi$.

In [7, Theorem 2.3] Kovár has obtained some important conditions under which a topology τ is doubly self-dual (that is, $\tau = \tau^{dd}$). Using condition (2) in [7, Theorem 2.3] Kovár [8, Example 2.1] has given the following example which may illustrate the important and interesting relationship between the Scott topology and compactness in terms of dual.

Example 2.1. Let L be a complete lattice (especially, a frame). Then for $(L, \omega(L))$, we have $\Phi^d = Q((L, \omega(L))) = \sigma(L)^c$ and $\omega(L)^d = \sigma(L)$ (cf. Lemma 4.1 in this note). Let $\Gamma = \{\uparrow x : x \in L\}$. Then Γ is a closed subbasis of $(L, \omega(L))$ and is up-compact. It is easy to see that Γ is Φ^d-down-conservative. By [7, Theorem 2.3], we have that $\omega(L) = \omega(L)^{dd}$ and hence $\sigma(L)^d = \omega(L)$ by $\omega(L)^d = \sigma(L)$. It follows that $\omega(L)$ and $\sigma(L)$ are duals for each other.