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Diagonalizations of dense families

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Weakly Hurewicz
Weakly Menger

ABSTRACT

We develop a unified framework for the study of classic and new properties involving diagonalizations of dense families in topological spaces. We provide complete classification of these properties. Our classification draws upon a large number of methods and constructions scattered in the literature, and on several novel results concerning the classic properties.

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$\mathsf{S}_1(\mathrm{O},\mathrm{D})$
Weakly C″
$\mathsf{S}_1(\mathcal{O},\mathcal{D})$
Weakly Rothberger

1. Introduction

The following diagonalization prototypes are ubiquitous in the mathematical literature (see, e.g., the surveys [29,19,31]):

$\mathsf{S}_1(\mathscr{A},\mathscr{B})$: For all $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathscr{A}$, there are $U_1\in\mathcal{U}_1, U_2\in\mathcal{U}_2,\ldots$ such that $\{U_n\colon n\in\mathbb{N}\}\in\mathscr{B}$.
$\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$: For all $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathscr{A}$, there are finite $\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\ldots$ such that $\bigcup_n\mathcal{F}_n\in\mathscr{B}$.

The papers [25,18] have initiated the simultaneous consideration of these properties in the case where $\mathscr{A}$ and $\mathscr{B}$ are important families of open covers of a topological space X. This unified study of topological properties, that were previously studied separately, had tremendous success, some of which were surveyed in the above-mentioned surveys. The field of *selection principles* is growing rapidly, and dozens of new papers appeared since these survey articles were published. The purpose of the present paper is to initiate a similar program for the case where $\mathscr{A}$ and $\mathscr{B}$ are dense families, as we now define.

Definition 1.1. Let X be a topological space. A family $\mathcal{U}\subseteq P(X)$ is a *dense family* if $\bigcup\mathcal{U}$ is a dense subset of X. A family $\mathcal{U}\subseteq P(X)$ is in:

D: if $\mathcal{U}$ is dense;
$\mathrm{D_o}$: if $\mathcal{U}$ is dense and all members of $\mathcal{U}$ are open; and
O: if $\mathcal{U}$ is an open cover of X.

In other words, $\mathcal{U}$ is a dense family if each open set in X intersects some member of $\mathcal{U}$. Note that

$$\mathrm{O}\subseteq\mathrm{D_o}\subseteq\mathrm{D}.$$

Every element of D is refined by a dense family of singletons. It follows, for example, that $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ is equivalent to the following property, studied under various names in the literature (see Table 1 below):

For each sequence A_n, $n\in\mathbb{N}$, such that $\overline{A_n}=X$ for all n, there are finite sets $F_n\subseteq A_n$, $n\in\mathbb{N}$, such that $\overline{\bigcup_n F_n}=X$.

We study all properties $\mathsf{S}(\mathscr{A},\mathscr{B})$ for $\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\}$ and $\mathscr{A},\mathscr{B}\in\{\mathrm{O},\mathrm{D_o},\mathrm{D}\}$, by making use of their inter-connections. This approach is expected to have impact beyond these properties, not only concerning properties that imply or are implied by the above-mentioned properties (e.g., the corresponding game-theoretic properties), but also concerning formally unrelated properties that have a similar flavor.

The properties we are studying here were studied in the literature under various, sometimes pairwise incompatible, names. Examples are given in Table 1 below, with some references. We do not give references for $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, because there are hundreds of them. Instead, we refer to the above-mentioned surveys. In this table, by *obsolete* we mean that nowadays the name stands for another property.

A topological space is *$\mathscr{A}$-Lindelöf* ($\mathscr{A}\in\{\mathrm{D},\mathrm{D_o},\mathrm{O},\ldots\}$) if each member of $\mathscr{A}$ contains a countable member of $\mathscr{A}$. If X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{A})$, then X is $\mathscr{A}$-Lindelöf. Thus, $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ spaces are Lindelöf, $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ spaces are separable, and $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D})$ spaces are $\mathrm{D_o}$-Lindelöf, or equivalently, c.c.c. (i.e., such that every

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