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Proper shape over finite coverings

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ABSTRACT

A proper shape is presented using an intrinsic definition and only finite coverings of open sets with compact boundary. For locally compact separable metric spaces with compact spaces of quasicomponents, it is shown: If X and Y have the same proper shape over finite coverings then their end points compactifications FX and FY have the same shape.

1. Introduction

Proper shape theory is introduced in the papers of Ball and Sher. One of the main results is the following theorem: If X and Y have the same proper shape, then their end point compactifications FX and FY have the same shape [2].

The aim of this paper is to introduce an intrinsic definition of proper shape using only finite coverings consisting of open sets with compact boundaries. The main theorem about the shape of end point compactifications remains the same with the new definition.

First we repeat the main notions about functions continuous up to a covering needed for the intrinsic definition of shape. About the other definitions of shape we refer to books [8] and [9].

For collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, $\mathcal{U} \prec \mathcal{V}$ means that $\mathcal{U}$ refines $\mathcal{V}$, i.e., each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. By a covering we understand a covering consisting of open sets.

Definition 1.1. Let X, Y be spaces, and $\mathcal{V}$ a covering of Y. The function $f : X \to Y$ is *$\mathcal{V}$-continuous*, if for any $x \in X$, there exists a neighborhood U of x, such that $f(U) \subseteq V$ for some member $V \in \mathcal{V}$.

(The family of all such U form a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}$-continuous, if there exists $\mathcal{U}$ such that $f(\mathcal{U}) \prec \mathcal{V}$.)

Definition 1.2. Two $\mathcal{V}$-continuous functions $f, g : X \to Y$ are *homotopic* if there exists a function $F : X \times I \to Y$ such that:

(1) $F : X \times I \to Y$ is $\operatorname{st}\mathcal{V}$-continuous.
(2) There exists a neighborhood $N = [0, \varepsilon) \cup (1 - \varepsilon, 1]$ of $\{0, 1\}$ in I, such that $F|_{X \times N}$ is $\mathcal{V}$-continuous.
(3) $F(x, 0) = f(x)$, $F(x, 1) = g(x)$.

We denote this by $f \overset{\mathcal{V}}{\simeq} g$.

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The main notion required for the intrinsic definition of shape for compact metric spaces is the notion of proximate sequence [5] (see also, [4] and [10]).

Definition 1.3. A sequence (f_n) of functions $f_n : X \to Y$ is a *proximate sequence* from X to Y if for some cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of finite coverings, for all indices $m \geqslant n$, f_n and f_m are homotopic as $\mathcal{V}_n$-continuous functions (cofinal means that for any finite covering $\mathcal{V}$ there exists $\mathcal{V}_n$ such that $\mathcal{V}_n \prec \mathcal{V}$).

In this case we say that (f_n) is a proximate sequence over $(\mathcal{V}_n)$.

We mention that if (f_n) and (f'_n) are proximate sequences from X to Y, then there exists a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$, such that (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$.

Two proximate sequences $(f_n), (f'_n) : X \to Y$ are *homotopic* if for some cofinal sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of finite coverings, (f_n) and (f'_n) are proximate sequences over $(\mathcal{V}_n)$, and for all integers n, f_n and f'_n are homotopic as $\mathcal{V}_n$-continuous functions. This is an equivalence relation, and we write $(f_n) \sim (f'_n)$. The homotopy class is denoted by $[(f_n)]$.

In the paper [4], $[X, Y]_{\mathcal{V}}$ denotes the set of all $\mathcal{V}$-homotopy classes of $\mathcal{V}$-continuous functions $f : X \to Y$. The inverse system is formed in the category of sets and functions

$$\big([X, Y]_{\mathcal{V}},\ p_{\mathcal{V}\mathcal{V}'},\ \mathcal{V} \text{ a finite covering}\big),$$

where for $\mathcal{V} \succ \mathcal{V}'$, $p_{\mathcal{V}\mathcal{V}'}([f]_{\mathcal{V}'}) = [f]_{\mathcal{V}}$. The inverse limit of this inverse system is denoted by $\varprojlim_{\mathcal{V}}[X, Y]_{\mathcal{V}}$. Also, a bijection is established between $\varprojlim_{\mathcal{V}}[X, Y]_{\mathcal{V}}$ and the set of all shape morphisms from X to Y.

In the paper [5], it is shown that there is a bijection between the set $\varprojlim_{\mathcal{V}}[X, Y]_{\mathcal{V}}$ and the set of homotopy classes $[(f_n)]$ of proximate sequences $(f_n) : X \to Y$. It follows that there is a bijection between all shape morphisms from X to Y and the set of all homotopy classes $[(f_n)]$ of proximate sequences $(f_n) : X \to Y$.

If $(f_n) : X \to Y$ is a proximate sequence and $(g_n) : Y \to Z$ a proximate sequence over $(\mathcal{W}_n)$, we can choose a cofinal sequence of finite coverings $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ of Y such that $(f_n) : X \to Y$ is a proximate sequence over $(\mathcal{V}_n)$, and $g(\mathcal{V}_n) \prec \mathcal{W}_n$ for all integers. The *composition* of the proximate sequences $(f_n) : X \to Y$ and $(g_n) : Y \to Z$ is the proximate sequence $(h_n) = (g_n f_n)$.

Compact metric spaces and homotopy classes of proximate sequences form the *shape category*, i.e. isomorphic spaces in this category have the same shape [5].

To conclude we present another description of the notion of proximate sequences.

Theorem 1.4. *$(f_n) : X \to Y$ is a proximate sequence if and only if for any finite covering $\mathcal{V}$ of Y there exists an integer n_0 such that for all $n \geqslant n_0$, f_{n_0} and f_n are homotopic as $\mathcal{V}$-continuous functions.*

Proof. Suppose (f_n) is a proximate sequence over $(\mathcal{V}_n)$, and $\mathcal{V}$ is a finite covering of Y. There exists $\mathcal{V}_{n_0}$ such that $\mathcal{V}_{n_0} \prec \mathcal{V}$. Since (f_n) is a proximate sequence over $(\mathcal{V}_n)$, for all $n \geqslant n_0$ holds $f_n \overset{\mathcal{V}_{n_0}}{\simeq} f_{n_0}$. It follows $f_n \overset{\mathcal{V}}{\simeq} f_{n_0}$.

In the other direction, suppose that for any finite covering $\mathcal{V}$ of Y there exists an integer n_0 such that for all $n \geqslant n_0$, f_{n_0} and f_n are homotopic as $\mathcal{V}$-continuous functions.

If $\mathcal{W}_1 \succ \mathcal{W}_2 \succ \cdots$ is a cofinal sequence in the set of all finite coverings of Y, for the covering $\mathcal{W}_1$ here exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ holds $f_n \overset{\mathcal{W}_1}{\simeq} f_{n_1}$, and f_n is $\mathcal{W}_1$-continuous. Similarly, for $\mathcal{W}_2$ there exists $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that for all $n \geqslant n_2$ holds $f_n \overset{\mathcal{W}_2}{\simeq} f_{n_2}$. In this way we obtain a sequence of integers $n_1 < n_2 < \cdots$ such that for all $n \geqslant n_k$ holds $f_n \overset{\mathcal{W}_k}{\simeq} f_{n_k}$, and f_n is $\mathcal{W}_k$-continuous.

We put $\mathcal{V}_i = \{Y\}$ for $i \in \{1, \ldots, n_1 - 1\}$, $\mathcal{V}_i = \mathcal{W}_1$, for $i \in \{n_1, \ldots, n_2 - 1\}$, $\mathcal{V}_i = \mathcal{W}_2$, for $i \in \{n_2, \ldots, n_3 - 1\}, \ldots, \mathcal{V}_i = \mathcal{W}_k$, for $i \in \{n_k, \ldots, n_{k+1} - 1\}, \ldots$.

Then the sequence $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ is cofinal in the set of finite coverings of Y, and for all $m \geqslant n$ holds $f_n \overset{\mathcal{V}_n}{\simeq} f_m$. □

Theorem 1.5. *Two proximate sequences $(f_n), (g_n) : X \to Y$ are homotopic if and only if for any finite covering $\mathcal{V}$ of Y there exists an integer n_0 such that for all $n \geqslant n_0$, f_n and g_n are homotopic as $\mathcal{V}$-continuous functions.*

Proof. Suppose (f_n) and (g_n) are proximate sequences homotopic over $(\mathcal{V}_n)$, and $\mathcal{V}$ is a finite covering of Y. There exists $\mathcal{V}_{n_0}$ such that $\mathcal{V}_{n_0} \prec \mathcal{V}$. Since $\mathcal{V} \succ \mathcal{V}_{n_0} \succ \mathcal{V}_{n_0+1} \succ \cdots$ and $f_n \overset{\mathcal{V}_n}{\simeq} g_n$ for all $n \geqslant n_0$, it follows that $f_n \overset{\mathcal{V}}{\simeq} g_n$ for all $n \geqslant n_0$.

In the other direction, suppose that for any finite covering $\mathcal{V}$ of Y there exists an integer n_0 such that for all $n \geqslant n_0$, f_n and g_n are homotopic as $\mathcal{V}$-continuous functions.

Suppose (f_n) and (g_n) are proximate sequences over $\mathcal{W}_1 \succ \mathcal{W}_2 \succ \cdots$. For each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $f_n \overset{\mathcal{W}_k}{\simeq} g_n$ for every $n \geqslant n_k$. We can choose $n_k < n_{k+1}$, for all $k \in \mathbb{N}$.

We put $\mathcal{V}_i = \{Y\}$ for $i \in \{1, \ldots, n_1 - 1\}$, $\mathcal{V}_i = \mathcal{W}_1$, for $i \in \{n_1, \ldots, n_2 - 1\}$, $\mathcal{V}_i = \mathcal{W}_2$, for $i \in \{n_2, \ldots, n_3 - 1\}$, $\ldots$, $\mathcal{V}_i = \mathcal{W}_k$, for $i \in \{n_k, \ldots, n_{k+1} - 1\}$, $\ldots$.

Then (f_n) and (g_n) are proximate sequences over $\mathcal{V}_1 \succ \mathcal{V}_2 \succ \cdots$ and $f_n \overset{\mathcal{W}_n}{\simeq} g_n$ for all $n \in \mathbb{N}$. □