Annals of Pure and Applied Logic 168 (2017) 1–18

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Products of Menger spaces: A combinatorial approach

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ARTICLE INFO

Article history:
Received 19 May 2016
Accepted 16 August 2016
Available online 24 August 2016

MSC:
primary 54D20
secondary 03E17

Keywords:
Menger property
Hurewicz property
Concentrated sets
Bi-unbounded sets
Reaping number
Scales

ABSTRACT

We construct Menger subsets of the real line whose product is not Menger in the plane. In contrast to earlier constructions, our approach is purely combinatorial. The set theoretic hypothesis used in our construction is far milder than earlier ones, and holds in almost all canonical models of set theory of the real line. On the other hand, we establish productive properties for versions of Menger's property parameterized by filters and semifilters. In particular, the Continuum Hypothesis implies that every productively Menger set of real numbers is productively Hurewicz, and each ultrafilter version of Menger's property is strictly between Menger's and Hurewicz's classic properties. We include a number of open problems emerging from this study.

1. Introduction

A topological space X is *Menger* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of the space X, there are finite subsets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ whose union forms a cover of the space X. This property was introduced by Karl Menger [17], and reformulated as presented here by Witold Hurewicz [11]. Menger's property is strictly between σ-compact and Lindelöf. Now a central notion in topology, it has applications in a number of branches of topology and set theory. The undefined notions in the following example, which are available in the indicated references, are not needed for the remainder of this paper.

Example 1.1. Menger spaces form the most general class for which a positive solution of the D-space problem is known [2, Corollary 2.7], and the most general class for which a general form of Hindman's Finite Sums

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http://dx.doi.org/10.1016/j.apal.2016.08.002
0168-0072/

Theorem holds [27]. In set theory, Menger's property characterizes filters whose Mathias forcing notion does not add dominating functions [9].

Menger's property is hereditary for closed subsets and continuous images. By a classic result of Todorčević there are, provably, Menger spaces X and Y such that the product space $X \times Y$ is not Menger [24, §3]. It remains open whether there are, provably, such examples in the real line, or even just metrizable examples [25, Problem 6.7]. This problem, proposed by Scheepers long ago, resisted tremendous efforts thus far.

For brevity, sets of real numbers are called here *real sets*.[1] An uncountable real set is *Luzin* if its intersection with every meager (Baire first category) set is countable. Assuming the Continuum Hypothesis, there are two Luzin sets whose product is not Menger [12, Theorem 3.7]. An uncountable real set X is *concentrated* if it has a countable subset D such that the set $X \setminus U$ is countable for every open set U containing D. Every Luzin set is concentrated, and every concentrated set has Menger's property. This approach extends to obtain similar examples using a set theoretic hypothesis about the meager sets that is weaker than the Continuum Hypothesis [21, Theorem 49]. Later methods [29, Theorem 9.1] were combined with reasoning on meager sets to obtain examples using another portion of the Continuum Hypothesis [19, Theorem 3.3].

We introduce a purely combinatorial approach to products of Menger sets. We obtain examples using hypotheses milder than earlier ones, as well as examples using hypotheses that are incompatible with the Continuum Hypothesis. To this end, we introduce the key notion of *bi-$\mathfrak{d}$-unbounded set*, and determine the limits on its possible existence. We extend these results to variations of Menger's property parameterized by filters and semifilters (defined below). For a semifilter S, we introduce the notion of *S-scale*. These scales provably exist, and capture a number of distinct special cases used in earlier works.

The second part of the paper, beginning with Section 5, establishes provably productive properties among semifilter-parameterized Menger properties. If S is an *ultrafilter*, then every S-scale gives rise to a real set that is productively S-Menger. We deduce that each of these variations of Menger's property is strictly between Hurewicz's and Menger's classic properties.

The last section includes a discussion of related results and open problems suggested by this study.

2. Products of Menger sets

Towards a combinatorial treatment of the questions discussed here, we identify the Cantor space $\{0,1\}^{\mathbb{N}}$ with the family $\mathrm{P}(\mathbb{N})$ of all subsets of the set $\mathbb{N}$. Since the Cantor space is homeomorphic to Cantor's set, every subspace of the space $\mathrm{P}(\mathbb{N})$ is considered as a real set.

The space $\mathrm{P}(\mathbb{N})$ splits into two important subspaces: the family of infinite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{\infty}$, and the family of finite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{<\infty}$. We identify every set $a \in [\mathbb{N}]^{\infty}$ with its increasing enumeration, an element of the Baire space $\mathbb{N}^{\mathbb{N}}$. Thus, for a natural number n, $a(n)$ is the n-th element in the increasing enumeration of the set a. This way, we have $[\mathbb{N}]^{\infty} \subseteq \mathbb{N}^{\mathbb{N}}$, and the topology of the space $[\mathbb{N}]^{\infty}$ (a subspace of the Cantor space $\mathrm{P}(\mathbb{N})$) coincides with the subspace topology induced by $\mathbb{N}^{\mathbb{N}}$. This explains some of the elementary assertions made here; moreover, notions defined here for $[\mathbb{N}]^{\infty}$ are often adaptations of classic notions for $\mathbb{N}^{\mathbb{N}}$. Depending on the interpretation, points of the space $[\mathbb{N}]^{\infty}$ are referred to as sets or functions.

For functions $a, b \in [\mathbb{N}]^{\infty}$, we write $a \leq b$ if $a(n) \leq b(n)$ for all natural numbers n, and $a \leq^* b$ if $a(n) \leq b(n)$ for almost all natural numbers n, that is, the set of exceptions $\{\, n : b(n) < a(n) \,\}$ is finite. We follow the convention that *bounded* means *has an upper bound* in the ambient superset.

[1] The term *real set* is a natural extension of the standard notions *real number*, *real matrix*, *real function*, etc., and should be understood as a convenient abbreviation. It does not imply that other sets are less "real".