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Effective results on a fixed point algorithm for families of nonlinear mappings

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ABSTRACT

We use proof mining techniques to obtain a uniform rate of asymptotic regularity for the instance of the parallel algorithm used by López-Acedo and Xu to find common fixed points of finite families of k-strict pseudocontractive self-mappings of convex subsets of Hilbert spaces. We show that these results are guaranteed by a number of logical metatheorems for classical and semi-intuitionistic systems.

1. Introduction

The class of k-strict pseudocontractions was introduced and studied by Browder and Petryshyn in [1] in the context of Hilbert spaces. (All Hilbert spaces and inner product spaces considered here will be over the real field.) If H is a Hilbert space, $C \subseteq H$ is a convex subset and $k \in [0, 1)$, then a mapping $T : C \to H$ is called a **k-strict pseudocontraction** if for all $x, y \in C$ we have that:

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2. \tag{1}$$

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If we set $k := 0$ in the above inequality, we obtain the condition $\|Tx - Ty\| \leq \|x - y\|$, which states that the mapping T is nonexpansive. The search of algorithms for finding fixed points of self-mappings of subsets of metric spaces belonging to various established classes has been a longstanding research program in the field of nonlinear analysis.

These algorithms are usually iterative in their nature – one begins from an arbitrary starting point and proceeds to repeatedly apply to it a transformation which is derived from the operator T. In this way, the algorithm produces a sequence $(x_n)_{n\in\mathbb{N}} \subseteq H$. Frequently, the first major intermediate step of the convergence proof is a result of *asymptotic regularity* – that is, a statement of the form

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

In this case, we also say that $(x_n)_n$ is an *approximate fixed point sequence* for T. Afterwards, using demiclosedness principles and/or various hypotheses of compactness, followed by considerations regarding the geometry of the space in question, one may obtain that the sequence $(x_n)_{n\in\mathbb{N}} \subseteq H$ converges, weakly or strongly, to a fixed point of T.

Proof mining is a research program introduced by U. Kohlenbach in the 1990s ([9] is a comprehensive reference, while a recent survey is [10]), which aims to obtain explicit quantitative information (witnesses and bounds) from proofs of an apparently ineffective nature. This paradigm in applied logic has successfully led so far to obtaining some previously unknown effective bounds, primarily in nonlinear analysis and ergodic theory. A large number of these are guaranteed to exist by a series of logical metatheorems which cover general classes of bounded or unbounded metric structures – see [8,2,3]. As an example of such a piece of quantitative information, let us define a *rate of convergence* for a real-valued sequence $(a_n)_{n\in\mathbb{N}}$ that has the limit $a \in \mathbb{R}$ to be a function $\Theta : (0,\infty) \to \mathbb{N}$ such that for all $\varepsilon > 0$ and all $n \geq \Theta(\varepsilon)$, we have that $\|a_n - a\| \leq \varepsilon$. If the real-valued sequence is the sequence $(\|x_n - Tx_n\|)$ introduced before, then Θ is called a *rate of T-asymptotic regularity* for the sequence (x_n). General metatheorems of proof mining usually guarantee that from the methods of proof used in nonlinear analysis one can obtain ("extract") computable rates of asymptotic regularity for the usual iterative schemas – see, for example, [12,13]. In particular, one such recent result of Ivan and Leuştean [4], analyzing a proof by Marino and Xu [15], states that for the Krasnoselski iteration of a single k-strict pseudocontractive self-mapping T of a convex subset of a Hilbert space, there exists such a computable rate of T-asymptotic regularity that is quadratic in $\frac{1}{\varepsilon}$.

That being said, such an area of investigation closely related to the kind of iterative algorithms mentioned before has been the problem of finding a common fixed point of a (finite or infinite) family $(T_i)_i$ of self-mappings of a subset C like above. An iterative scheme that is useful in the case of a finite family $(T_i)_{1\leq i\leq N}$ is the *parallel algorithm*, defined as follows. Let x be in C and $(t_n)_{n\in\mathbb{N}} \subseteq (0,1)$. For each $i \in \{1,\ldots,N\}$, let $(\lambda_i^{(n)})_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that, for any $n \in \mathbb{N}$:

$$\sum_{i=1}^{N} \lambda_i^{(n)} = 1.$$

Write, for all $n \geq 0$:

$$A_n := \sum_{i=1}^{N} \lambda_i^{(n)} T_i.$$

Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by:

$$x_0 := x,$$
$$x_{n+1} := t_n x_n + (1 - t_n) A_n x_n.$$