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Proof complexity of intuitionistic implicational formulas

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A R T I C L E I N F O

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We study implicational formulas in the context of proof complexity of intuitionistic propositional logic (**IPC**). On the one hand, we give an efficient transformation of tautologies to implicational tautologies that preserves the lengths of intuitionistic extended Frege (*EF*) or substitution Frege (*SF*) proofs up to a polynomial. On the other hand, *EF* proofs in the implicational fragment of **IPC** polynomially simulate full intuitionistic logic for implicational tautologies. The results also apply to other fragments of other superintuitionistic logics under certain conditions.

In particular, the exponential lower bounds on the length of intuitionistic *EF* proofs by Hrubeš (2007), generalized to exponential separation between *EF* and *SF* systems in superintuitionistic logics of unbounded branching by Jeřábek (2009), can be realized by implicational tautologies.

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1. Introduction

A major open problem in proof complexity is to show superpolynomial lower bounds on the lengths of proofs in Frege systems for classical propositional logic, or even stronger systems such as extended Frege. It turns out such lower bounds are easier to obtain for some non-classical logics: Hrubeš proved exponential lower bounds on the length of *EF* proofs[1] for certain modal logics and for intuitionistic logic [8–10]. Jeřábek [12] improved these results to an exponential separation between *EF* and *SF* systems for all superintuitionistic (and transitive modal) logics of unbounded branching. See also [3,4] for earlier work on the proof complexity of intuitionistic logic, including conditional lower bounds.

Known lower bounds on proof systems for non-classical logics crucially rely on variants of the feasible disjunction property, serving a similar role as feasible interpolation does in weak proof systems for classical logic (cf. [15]). Consequently, the lower bounds are proved for tautologies that involve disjunction in an

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[1] The results are formulated in [8–10] as lower bounds on the number of lines in Frege proofs, however, this is essentially the same measure as the length of extended Frege proofs.

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essential way, and one might get the impression that this is unavoidable—perhaps the implicational fragment of **IPC**, or other disjunction-free fragments, behave differently from the full logic.

The purpose of this paper is to show that this dependence on disjunctions is just an artifact of the proofs: the implicational fragment of intuitionistic logic is, after all, essentially equivalent to the full logic with respect to the lengths of proofs. We will demonstrate this by means of two kinds of results: first, *tautologies* can be brought into a form avoiding unwanted connectives (such as disjunction) while preserving their hardness for intuitionistic extended Frege and related systems; second, unwanted connectives can be eliminated from intuitionistic extended Frege *proofs* except for subformulas of the tautology being proved. We include several results of both kinds with varying assumptions.

Elimination results of the first kind are the topic of Section 3. On the one hand, in Theorem 3.6 we present a method that makes tautologies *mostly* implicational with certain disjunctions and $\bot$ left, and preserves (up to a polynomial) the size of F, EF, and SF proofs in *arbitrary* superintuitionistic logics; in particular, the tautologies with exponential EF lower bounds from [9,12] can be made purely implicational in this way. On the other hand, in Theorems 3.8 and 3.9 we show how to eliminate *all* disjunctions (and/or $\bot$) from tautologies while preserving the lengths of EF and SF proofs in logics whose proper axioms do not contain disjunctions ($\bot$, respectively).

Elimination results of the second kind come in Section 4. In Theorem 4.3, we show that if the proper axioms of a logic L do not contain disjunction (or $\bot$), we can efficiently eliminate disjunctions ($\bot$, resp.) from L-EF proofs, except for those that appear in the final tautology. However, the argument may introduce conjunctions, and we address this in subsequent results: in Corollary 4.15 and Theorem 4.17, we show how to eliminate conjunctions from EF-proofs under some conditions on the logic and its axiom system; in Theorem 4.5, we show how to eliminate $\bot$ from proofs without introducing $\wedge$ or other connectives, again under certain conditions on the logic. We also develop a monotone version of the negative translation (Proposition 4.19), which we use in an ad hoc argument that the above-mentioned implicational versions of the tautologies used in [12] to separate EF and SF have short implicational **IPC**-SF proofs (Theorem 4.22).

A few concluding remarks and open problems are mentioned in Section 5.

In order to show the limitations of our methods, the appendices include some negative results that may be of independent interest. Proposition A.5, originally due to Wroński [18], shows that in general, the implicational fragment of a superintuitionistic logic $L = \mathbf{IPC} + \Phi$ with Φ an implicational axiom may not be axiomatized by Φ over the implicational fragment of **IPC**, and similarly for other combinations where the target fragment omits conjunction. Appendix B presents certain exponential lower bounds on the size of formulas in fragments of intuitionistic logic, and a linear lower bound on implication nesting depth.

2. Preliminaries

We refer to Chagrov and Zakharyaschev [5] and Krajíček [14] for general information on superintuitionistic logics and classical Frege systems and their extensions, respectively. For Frege and friends in superintuitionistic logics, we will use the notation and basic results from Jeřábek [12]; we include more details below, as we need to generalize the set-up to fragments (e.g., implicational) of si logics, which were not treated in [12].

If L is a language,[2] a *proof system for* L is a polynomial-time computable function $P(x)$ whose range is L. If $P(x) = y$, the string x is called a P-*proof* of y. A proof system P *polynomially-simulates* or *p-simulates* a proof system Q, denoted $Q \leq_p P$, if there is a poly-time function f such that $Q(x) = P(f(x))$ for all proofs x. Proof systems P and Q are *p-equivalent*, written $P \equiv_p Q$, if $P \leq_p Q \leq_p P$.

A *propositional language* is a set C of connectives, each given a finite arity. (That is, formally, a language is a mapping $ar\colon C \to \omega$.) Let Form_C denote the set of C-*formulas*, built from propositional variables p_i, $i \in \omega$, using connectives from C. We will also denote variables by other lowercase Latin letters for

[2] In the sense of the theory of computation, i.e., an arbitrary set of strings over a finite alphabet.

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