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Essential and density topologies of continuous domains

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We introduce and study two topologies in order to provide a topological interpretation of bases in domain theory. The key finding is that, in a continuous domain, bases correspond exactly to dense sets of one of these new topologies. Moreover, we provide a topological interpretation for several properties of the bases, as well as novel characterizations of algebraic domains.

1. Introduction

This paper represents a continuation of our previous algebraic and topological approaches [2–4]. Here we work with directed complete partial order sets (dcpo) defined as partial order sets (posets) with the property that every directed subset has an upper bound. If we look at such a dcpo as a domain of information, then $x \leq y$ means that y contains all the information of x (and possibly additional information). An example of directed set is provided by increasing sequences $x_0 \leq x_1 \leq \dots$ called chains; such a sequence can be interpreted as a computational process in which at each step more and more information is obtained and stored in the elements of the sequence. The upper bound of this sequence is an element containing all the information obtained by the computational process. Even computational processes could be formalized as directed sets. It is natural for a process to work with a certain redundancy; however, we are interested in essential information. We say that x is an essential part of y if any formalized computation process generating the information y also generates the information contained in x. An element x is called compact if it is an essential part of itself.

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A dcpo X is called a continuous domain if every element $x \in X$ can be obtained from its essential parts, meaning that the set of its essential parts is directed and its upper bound is x. Often, deriving the element x does not require all its essential parts, but only a subset of them. Thus, a subset $B \subseteq X$ is called a basis for X if every element $x \in X$ can be obtained from essential parts belonging to B. A continuous domain is called algebraic if the set of its compact elements forms a basis. Since the upper bound can be seen as a limit, a basis can be interpreted as a dense set in X. A question naturally arises, namely whether there is a topology on X such that its bases are dense in a topological sense. This article provides an answer to this question. We build a topology on a continuous domain X such that the bases of X coincide with the dense sets of this topology. In this way, many results of algebraic nature receive a topological interpretation. On the other hand, a topological treatment of domains bases can lead to new interesting results.

We define an intermediate topology over a dcpo that we call the e-topology. It consists of what we call *e-open sets* containing the essential parts of their elements. The smallest common refinement of the Scott topology and the e-topology on a dcpo X is called the *density topology*, and it is denoted by ρ_X. The density topology is a Hausdorff topology; it is finer than the Lawson topology. In a continuous domain X, a subset is a basis if and only if it is a dense set in the density topology. Using this equivalence, we prove new characterizations for algebraic domains. Also, a continuous domain X is ω-continuous if and only if the associated topological space (X, ρ_X) is separable.

In any topological space, closed sets with empty interior coincide with the class of dense open sets; these sets are "negligible". A topological space is called a Baire space if every countable intersection of dense open sets is dense. Intuitively, this means that the space is "large", as it cannot be represented as a countable union of "negligible" subsets. Although the quality of a Baire space derives from a topological property, little is known about such spaces. The main results (having several applications) on Baire spaces state that "any complete metric space is second category" and "any locally compact space is a Baire space" [9]. We show that for any algebraic domain X, the associated topological space (X, ρ_X) is a Baire space.

2. Essential topology of continuous domains

For a nonempty set X we denote by $\mathcal{P}(X)$ the set of all subsets of X, and by $\mathcal{P}_{fin}(X)$ the set of all finite subsets of X. Let $(X, \leq)$ be a partially ordered set. For $A \subseteq X$, we denote by $\uparrow A$ the upper set of A, i.e. $\{x \in X \mid \exists a \in A, a \leq x\}$. In a similar way, the set $\downarrow A$ is the lower set of A, i.e. $\{x \in X \mid \exists a \in A, x \leq a\}$. For $x \in X$, the set $\uparrow \{x\}$ is denoted by $\uparrow x$, and the set $\downarrow \{x\}$ is denoted by $\downarrow x$. We denote by $\underline{\downarrow}\, x$ the set $\downarrow x \setminus \{x\}$. A subset A of X is an upper set if $\uparrow A = A$, and a lower set if $\downarrow A = A$.

A partially ordered set in which every chain has a supremum is a chain-complete partial order (ccpo). A partially ordered set in which every directed subset has a supremum is a directed-complete partial order (dcpo).

Let $(X, \leq)$ be a dcpo.

Definition 1. For any $x, y \in X$ we say that "x approximates y" or "x is an essential part of y" (and write $x \ll y$) if for all directed subsets A of X with $y \leq supA$, there is $a \in A$ such that $x \leq a$. We say that x is compact if it is an essential part of itself. The set of compact elements of X is denoted by $K(X)$.

For $x \in X$, we denote by $\twoheaddownarrow x$ the set $\{y \in X \mid y \ll x\}$, and by $\twoheaduparrow x$ the set $\{y \in X \mid x \ll y\}$. Then the set $\twoheaddownarrow A$ is equal to $\bigcup\limits_{x \in A} \twoheaddownarrow x$, and $\twoheaduparrow A$ is equal to $\bigcup\limits_{x \in A} \twoheaduparrow x$.

Definition 2.

– A subset B of a dcpo X is a basis for X if for every element x of X, the set $B \cap \twoheaddownarrow x$ contains a directed subset with supremum x.