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Limit spaces with approximations

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ABSTRACT

Abstracting from a presentation of the density theorem for the hierarchy Ct(ρ) of countable functionals over $\mathbb{N}$ given by Normann in [12], we define two subcategories of limit spaces, the limit spaces with approximations, and the limit spaces with general approximations, for both of which a density theorem holds directly. We show that these categories are cartesian closed, and we give examples of such limit spaces and of density theorems for hierarchies of functionals over them. Most of our main proofs are within Bishop's informal system of constructive mathematics BISH. In a limit space with (general) approximations the approximation functions are given beforehand as an internal part of the structure under study. In this way limit spaces with (general) approximations form a constructive approach to abstract limit spaces, reflecting at the same time the central idea of Normann's Program of Internal Computability.

1. Introduction

In this paper we generalize Normann's notion of the nth approximation of a functional in the typed hierarchy Ct(ρ) over $\mathbb{N}$, defining two subcategories of the category of limit spaces **Lim**, the category **Appr** of limit spaces with approximations, and the category **Gappr** of limit spaces with general approximations. These limit spaces, which are studied here constructively, realize in a direct way Normann's notion of internal computability.

Normann formulated the distinction between internal and external computability over a mathematical structure already in [11], and initiated, what we call, a *Program of Internal Computability* in [12–15].

According to [11], "the *internal* concepts [of computability] must grow out of the structure at hand, while *external* concepts may be inherited from computability over superstructures via, for example, enumerations, domain representations, or in other ways". Normann's motivation behind an internal approach in general computability is technical (see his results in [14]), conceptual (an associate-free description of Ct(ρ)) and practical, since within a weaker concept of computability (if an object is internally computable, then it is

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also externally computable, but not necessarily the converse) "the weaker tools we use to obtain a result, the more extra knowledge can be obtained from the process of obtaining the result" ([13], p. 305). Normann found suitable for his study of internal computability over a mathematical structure the framework of limit spaces. As he mentions in [12], p. 474, he finds it "useful to see how far we can get towards constructing an effective infrastructure on such spaces without introducing superstructures and imposing external notions of computability on the given structures ... One way to create a useful part of an infrastructure will be to isolate a dense subset that in some way is effectively dense.".

As we show in this paper, such dense sets are very direct to find in limit spaces with (general) approximations.[1] Although Normann is working in a classical framework, here we study these limit spaces within Bishop's informal system of constructive mathematics BISH (see [1–3]). When a proposition is proved with the use of non-constructive methods we write that it is in CLASS, the classical extension of BISH. In order to fix our notation and be self-contained we include some necessary definitions and facts.

2. Basic definitions and facts

A *limit space* is a structure $\mathbb{L} = (X, \lim)$, where X is an inhabited set, and $\lim \subseteq X \times X^{\mathbb{N}}$ is a relation satisfying the following conditions: (i) if $x \in X$ and (x) denotes the constant sequence x, then $\lim(x, (x))$, (ii) if $\mathcal{S}$ denotes the set of all elements of the Baire space $\mathcal{N}$ which are strictly monotone, then[2] $\forall_{\alpha \in \mathcal{S}}(\lim(x, x_n) \rightarrow \lim(x, x_{\alpha(n)}))$, and (iii) if $x \in X$ and $x_n \in X^{\mathbb{N}}$, then $\forall_{\alpha \in \mathcal{S}} \exists_{\beta \in \mathcal{S}}(\lim(x, x_{\alpha(\beta(n))})) \rightarrow \lim(x, x_n)$. In the literature condition (iii) is usually written as (iii)$'$ $\neg(\lim(x, x_n)) \rightarrow \exists_{\alpha \in \mathcal{S}} \forall_{\beta \in \mathcal{S}}(\neg\lim(x, x_{\alpha(\beta(n))}))$, but we prefer to have the intuitionistically stronger condition (iii) right from the start.[3] If $\forall_{x,y \in X} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \rightarrow \lim(y, x_n) \rightarrow x = y)$, then the limit space has the *uniqueness property*.[4] A limit space has the *weak uniqueness property*, if $\forall_{x,y \in X}(\lim(x, y) \rightarrow x = y)$. One can show classically,[5] that there exists a limit space with the weak uniqueness property which does not have the uniqueness property.

A limit space induces a natural topology $\mathcal{T}_{\lim}$, the Birkhoff–Baer topology, according to which a set $\mathcal{O} \subseteq X$ is lim-*open*, if $\forall_{x \in \mathcal{O}} \forall_{x_n \in X^{\mathbb{N}}}(\lim(x, x_n) \rightarrow \mathrm{ev}(x_n, \mathcal{O}))$, where, if $A \subseteq X$, we define $\mathrm{ev}(x_n, A) :\leftrightarrow \exists_{n_0} \forall_{n \geq n_0}(x_n \in A)$. A topological space $(X, \mathcal{T})$ induces a limit space $(X, \lim_{\mathcal{T}})$, where $\lim_{\mathcal{T}}(x, x_n) :\leftrightarrow x_n \overset{\mathcal{T}}{\rightarrow} x$, and $x_n \overset{\mathcal{T}}{\rightarrow} x$ denotes the convergence of x_n to x w.r.t. the topology $\mathcal{T}$. If $\mathbb{L}$ is a limit space, it is direct to see that $\lim(x, x_n) \rightarrow (x_n \overset{\mathcal{T}_{\lim}}{\rightarrow} x)$ i.e., $\lim \subseteq \lim_{\mathcal{T}_{\lim}}$. A limit space is called *topological*, if $\lim = \lim_{\mathcal{T}_{\lim}}$. It is also direct that $\mathcal{T} \subseteq \mathcal{T}_{\lim_{\mathcal{T}}}$. A topological space is called *sequential*, if $\mathcal{T} = \mathcal{T}_{\lim_{\mathcal{T}}}$.

A set $F \subseteq X$ is called lim-*closed*, if it is the complement of a lim-open set, and in CLASS we have that F is lim-closed iff $\forall_{x \in X} \forall_{x_n \in X^{\mathbb{N}}}(x_n \subseteq F \rightarrow \lim(x, x_n) \rightarrow x \in F)$. One can show that if $\mathbb{L}$ is topological and the induced topological space is 1st countable, the previous characterization of a lim-closed set can be carried out in BISH. A similar remark holds for other classical results within limit spaces.

A set $D \subseteq X$ is called lim-*dense*, if $\forall_{x \in X} \exists_{d_n \in D^{\mathbb{N}}}(\lim(x, d_n))$, while a limit space is called lim-*separable*, if there is a countable lim-dense subset of it. It is direct to show in BISH that if D is a lim-dense set, then D is dense in $(X, \mathcal{T}_{\lim})$, while one can show in CLASS that if D is dense in $(X, \mathcal{T})$, then it is not generally the case that D is $\lim_{\mathcal{T}}$-dense.[6]

[1] Limit spaces, as special case of weak limit spaces, have also been studied within Type-2 Theory of Effectivity (see the work of Schröder [19] and [20]), but from a non-constructive and an external computability point of view.

[2] If $(x_n)_n \in X^{\mathbb{N}}$ we write for simplicity $\lim(x, x_n)$ instead of $\lim(x, (x_n)_n)$, and $\lim(x, x)$ instead of $\lim(x, (x))$. If it is necessary, we write $\lim_n(x, x_n)$ to specify the convergence w.r.t. n. Usually one finds the notation $\lim_n x_n = x$ instead of $\lim(x, x_n)$.

[3] Menni and Simpson in [9] also use (iii) instead of (iii)$'$ in the definition of a limit space.

[4] A limit space with the uniqueness property is what Kuratowski calls in [8] an $\mathcal{L}^*$-space.

[5] All proofs not included here can be found in [17].

[6] E.g., the set of irrational numbers $\mathbb{I}$ is dense in $(\mathbb{R}, \mathcal{T}_{\mathrm{coc}})$, but it is not lim-dense in $(\mathbb{R}, \lim_{\mathcal{T}_{\mathrm{coc}}})$, where $\mathcal{T}_{\mathrm{coc}}$ is the cocountable topology on $\mathbb{R}$.