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Atomicity, coherence of information, and point-free structures

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We prove basic facts about the properties of atomicity and coherence for Scott information systems, and we establish direct connections between coherent information systems and well-known point-free structures.

1. Introduction

Domain theory has been a well-established branch of mathematics for several years now, one that exhibits a wide array of applications [1,5]. In particular it bears great significance regarding the denotational semantics of programming languages, which was historically one of the reasons that the theory emerged in the first place.

In [24], Dana Scott represents *Scott domains*, that is, pointed complete partial orders (cpo's) which are additionally consistently complete and algebraic, by *information systems*. These are supposed to structure atomic tokens of information according to their *consistency* and *entailment*, where entailment models deduction of information and preorders the carrier, so that the actual objects of the domain are then recovered as ideals. Scott's information systems have served as a natural approach to the domain-theoretic treatment of semantics, at least from the computer scientist's viewpoint, since they provide the means to discuss higher-type algorithms in a tangible way, namely in terms of their *finite approximations*: their tokens and the finite consistent sets of tokens that they consist of. In the context of information systems, the principle of

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finite support for computation finds one of its uttermost formulations: an algorithm is but a consistent and deductively closed collection of concrete, finite pieces of information. But there are also theoretical merits, as information systems are more basic than the domains they induce. In particular, properties of ideals reduce to properties of tokens and consistent sets, thus providing the possibility of elementary methods of argumentation. A prime example of this is the solution of domain equations up to identity [24,29,28], rather than up to isomorphism, as previous arguments could already show [25].

In our case, favoring the tangible nature of information systems is tied to the development of a constructive formal theory of partial computable functionals, one that should lend itself as naturally and intuitively as possible to an implementation in a proof assistant [7]. This objective motivates an in-depth study of information systems in their own sake, and leads to a bottom-up, constructive, and implementable redevelopment of domain theory for higher-type computability.

Helmut Schwichtenberg [22] started this redevelopment by employing a cartesian-closed class of information systems which feature *coherence* as well as *atomicity*, which technically reduce consistency and entailment, respectively, to binary predicates. Both of these properties, in one version or another, have proved crucial to various studies in denotational semantics. Already in the formative period of domain semantics, coherence came to the attention of Gordon Plotkin [16], while he was arguing for using cpo's instead of lattices and noticed that it is a quite omnipresent property in the usual domains of study; later, it became one of the key features of the standard model of Jean-Yves Girard's *linear logic* [6]. As for atomicity (also known as "linearity"), it is notably needed for the representation of stable domains [30], which are important in the study of the notorious notion of *sequentiality* [2], but also appears, again, in particular models of linear logic [4].

Focus and contributions of the paper

In this work we retain a top-down approach and present results [9] which relate, in a direct way, some of the point-free structures that have been put to successful use in the past by the community, to *coherent* information systems. The work has a rather cartographic flavor which we deem necessary in order to clearly understand the nature of information systems we have used in practice from a point-free viewpoint.

We begin in section 2 by recalling basic facts and observations concerning information systems. In section 3 we define the notions of atomicity and coherence and we show that atomic and coherent versions of information systems feature more ideals than the generic version. In section 4 we concentrate on point-free versions of coherence (atomicity, of apparently limited use in comparison to coherence, would be a subject for another text): we consider well-known point-free structures, namely domains, precusl's, and formal topologies, and impose appropriate coherence properties on them to show that they correspond to coherent information systems. Such correspondences naturally imply certain categorical equivalences, which we make explicit in the case of formal topologies, the less covered case of the three in the literature. In section 5 we gather some relevant notes and an outlook on future work.

2. Scott information systems

A *(Scott) information system* is a triple $\rho = (\mathsf{Tok}_\rho, \mathsf{Con}_\rho, \vdash_\rho)$, where Tok_ρ is a countable set of *tokens*, $\mathsf{Con}_\rho \subseteq \mathscr{P}_f(\mathsf{Tok}_\rho)$ is a collection of *consistent sets*, also called *(formal) neighborhoods* and $\vdash_\rho \subseteq \mathsf{Con}_\rho \times \mathsf{Tok}_\rho$ is an *entailment* relation, such that: consistency is reflexive and closed under subsets; entailment is reflexive and transitive; consistency propagates through entailment. Formally we have[1]

$$\{a\} \in \mathsf{Con},$$

[1] In general, we may drop the subscripts when we can afford it. We will typically use $a, b, \ldots$ for tokens and $U, V, \ldots$ for neighborhoods and finite sets.