Annals of Pure and Applied Logic 167 (2016) 770–793

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

A constructive manifestation of the Kleene–Kreisel continuous functionals

Martín Escardó *, Chuangjie Xu

University of Birmingham, UK

A R T I C L E I N F O

Article history:
Available online 20 April 2016

MSC:
03B15
03F50
03C90
03D65

Keywords:
Constructive mathematics
Uniform continuity
Fan functional
Sheaves
Kleene–Kreisel continuous functionals
Intuitionistic type theory

A B S T R A C T

We identify yet another category equivalent to that of Kleene–Kreisel continuous functionals. Reasoning constructively and predicatively, all functions from the Cantor space to the natural numbers are uniformly continuous in this category. We do not need to assume Brouwerian continuity axioms to prove this, but, if we do, then we can show that the full type hierarchy is equivalent to our manifestation of the continuous functionals. We construct this manifestation within a category of concrete sheaves, called C-spaces, which form a locally cartesian closed category, and hence can be used to model system T and dependent types. We show that this category has a *fan functional* and validates the uniform-continuity principle in these theories. Our development is within informal constructive mathematics, along the lines of Bishop mathematics. However, in order to extract concrete computational content from our constructions, we formalized it in intensional Martin-Löf type theory, in Agda notation, and we discuss the main technical aspects of this at the end of the paper.

1. Introduction

In a cartesian closed category with a natural numbers object ℕ, define the *simple objects* to be the least collection containing ℕ and closed under products and exponentials (function spaces). The simple objects of any such category give an interpretation of the simply typed lambda calculus and higher-type primitive recursion (the term language of Gödel's system T). The Kleene–Kreisel *continuous functionals*, or *countable functionals* [34,30], form a category equivalent to the full subcategory on the simple objects of any of the following categories, among others: (1) compactly generated topological spaces [34,16], (2) sequential topological spaces [16], (3) Simpson and Schröder's QCB spaces [3,16], (4) Kuratowski limit spaces [23], (5) filter spaces [23], (6) Scott's equilogical spaces [4], (7) Johnstone's *topological topos* [25]. See Normann [35] and Longley [28,29] for the relevance of Kleene–Kreisel spaces in the theory of higher-type computation.

* Corresponding author.
E-mail address: m.escardo@cs.bham.ac.uk (M. Escardó).

http://dx.doi.org/10.1016/j.apal.2016.04.011

Counterexamples include Hyland's *effective topos* [24] and the hereditary effective operations (HEO) [28], which give a second simple-type hierarchy, not discussed in this paper (see [28] for a discussion). A third type hierarchy, discussed here in connection with the continuous functionals, is the *full type hierarchy*, which is the full subcategory on the simple objects of the category of sets [34].

We work with a category of sheaves, analogous to the topological topos, and with a full subcategory of concrete sheaves [2], here called C-*spaces*, analogous to the limit spaces (Section 2). The C-spaces can be described as sets equipped with a suitable continuity structure, and their natural transformations can be regarded as continuous maps. The main contributions of this work are summarized as follows:

1. The simple C-spaces form a category equivalent to that of Kleene–Kreisel continuous functionals (Section 3.1).
   The proof here is non-constructive (as are the proofs of the above equivalences). But we claim that the C-spaces form a good substitute of the above categories of spaces for the purposes of constructive reasoning.
2. If we assume the Brouwerian axiom that all set-theoretic functions $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ are uniformly continuous, then we can show constructively that the full type hierarchy is equivalent to the Kleene–Kreisel continuous hierarchy within C-spaces (Section 3.2).
3. Without assuming Brouwerian axioms, we show constructively that the category of C-spaces has a *fan functional* $(\mathbf{2}^{\mathbb{N}} \to \mathbb{N}) \to \mathbb{N}$ that continuously calculates moduli of uniform continuity of maps $\mathbf{2}^{\mathbb{N}} \to \mathbb{N}$ (Section 4.1).
4. C-Spaces give a model of system T with a uniform-continuity principle (Section 4.2), expressed as the skolemization of
$$\forall f \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}.\ \exists m \in \mathbb{N}.\ \forall \alpha, \beta \in \mathbf{2}^{\mathbb{N}}.\ \alpha =_m \beta \Longrightarrow f\alpha = f\beta,$$
   where $\alpha =_m \beta$ stands for $\forall i < m.\ \alpha_i = \beta_i$, with the aid of a fan-functional constant.
5. C-Spaces give a model of dependent types with a uniform-continuity principle, expressed as a type via the Curry–Howard interpretation (Section 4.3):
$$\Pi(f \colon \mathbf{2}^{\mathbb{N}} \to \mathbb{N}).\ \Sigma(m \colon \mathbb{N}).\ \Pi(\alpha, \beta \colon \mathbf{2}^{\mathbb{N}}).\ \alpha =_m \beta \to f\alpha = f\beta$$
6. We give a constructive treatment of C-spaces (Section 2) suitable for development in a predicative intuitionistic type theory in the style of Martin-Löf [32], which we formalized in Agda notation [33,8] for concrete computational purposes, and whose essential aspects are discussed in Section 5.
   We stress, however, that in this paper we deliberately reason informally, along the lines of Bishop mathematics [6].

Among the above, (3) and (4), and part of (6), appeared in the preliminary conference version of this paper [40]. The other contributions, (1), (2), (5), regarding Kleene–Kreisel spaces and dependent types, and part of (6), regarding the formalization in predicative intuitionistic type theory, are new as far as we are aware, but of course there are connections with related work discussed below.

As mentioned above, our sheaf topos is closely related to Johnstone's topological topos. To build the topological topos, one starts with the monoid of continuous endomaps of the one-point compactification of the discrete natural numbers, and then takes sheaves for the canonical topology of this monoid considered as a category. The category of sequential topological spaces is fully embedded in the topological topos. The concrete sheaves are precisely the Kuratowski limit spaces, called *subsequential spaces* by Johnstone, because they are the subobjects of the sequential topological spaces. Working non-constructively, one can show that the topological topos has a fan functional and that it interprets uniform-continuity principles for both simple and dependent types.