Annals of Pure and Applied Logic 167 (2016) 794–805

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

# The intrinsic topology of Martin-Löf universes

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ARTICLE INFO

*Article history:*
Available online 28 April 2016

*MSC:*
03B15
03F50
03C90
03D65

*Keywords:*
Martin-Löf type theory
Universe
Topology
Rice's Theorem

ABSTRACT

A construction by Hofmann and Streicher gives an interpretation of a type-theoretic universe $U$ in any Grothendieck topos, assuming a Grothendieck universe in set theory. Voevodsky asked what space $U$ is interpreted as in Johnstone's topological topos. We show that its topological reflection is indiscrete. We also offer a model-independent, *intrinsic* or *synthetic*, description of the topology of the universe: It is a theorem of type theory that the universe is sequentially indiscrete, in the sense that any sequence of types converges to any desired type, up to equivalence. As a corollary we derive *Rice's Theorem for the universe*: it cannot have any non-trivial, decidable, extensional property, unless WLPO, the weak limited principle of omniscience, holds.



## 1. Introduction

A construction by Hofmann and Streicher [15] gives an interpretation of a type-theoretic universe $U$ in any Grothendieck topos, assuming a Grothendieck universe in set theory. Voevodsky asked us what space $U$ is interpreted as in Johnstone's topological topos [9]. We show that its topological reflection is indiscrete.

This answer is perhaps shocking at first sight: one would maybe expect a rather elaborate and interesting topology for the universe, but it turns out to be trivial in this model. Perhaps the topological topos, lacking univalence [16], falls short of giving an informative interpretation of the universe in type theory, or perhaps the Hofmann–Streicher interpretation of the universe is at fault.

None of these is the case. It is a theorem *of* type theory that the universe is *intrinsically indiscrete*: Any sequence $X_n$ of types in the universe converges to any desired type $X_\infty$, up to equivalence. We work with the notion of equivalence from homotopy type theory [16], denoted by $\simeq$. Type equivalence is logically equivalent to the existence of back and forth maps that pointwise compose to the identities, but is defined in a subtler way.

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http://dx.doi.org/10.1016/j.apal.2016.04.010

A convergent sequence is defined as a map from the *one-point compactification* $\mathbb{N}_\infty$ of the discrete type $\mathbb{N}$, constructed as the type of decreasing binary sequences. The idea is that the points $\underline{n} = 1^n 0^\omega$ are thought to converge to the point $\infty = 1^\omega$. This is in the spirit of *synthetic topology*, but we don't need to postulate any axiom, as opposed to the usual situations in synthetic topology [2] or other synthetic theories, to prove this.

The crucial observation is that the point at infinity is not detachable, without excluded middle or similar constructive taboo, from the finite points. In particular, we cannot define a function $\mathbb{N}_\infty \to X$ by case analysis on whether the argument is $\infty$ or not, as this amounts to the *weak limited principle of omniscience* WLPO [6]. We say that a sequence $x \colon \mathbb{N} \to X$ converges to a point $x_\infty : X$ if it extends to a function $\mathbb{N}_\infty \to X$ that maps $\infty$ to $x_\infty$. All functions automatically preserve limits of convergent sequences, or are sequentially continuous, by composition. If WLPO holds, then any sequence converges to any point, rendering all types indiscrete. From a topological point of view, WLPO violates traditional continuity axioms, and, from a computational point of view, it gives an oracle for solving the Halting Problem, which shows that WLPO fails to be constructive [4].

The above type-theoretic theorem then is that for any sequence $X : \mathbb{N} \to U$ of types and any type $X_\infty : U$, there is $A : \mathbb{N}_\infty \to U$ such that $A_{\underline{n}} \simeq X_n$ and $A_\infty \simeq X_\infty$, which can be formulated as saying that any sequence $X_n$ converges to any desired type $X_\infty$, up to equivalence. If the univalence axiom holds, then of course $X_n$ converges to $X_\infty$ on the nose, as the univalence axiom implies that equivalent types are equal. We do not assume univalence, but we do need to assume the axiom of functional extensionality (any two pointwise equal functions are equal, which is a consequence of univalence) to prove some basic properties of the type $\mathbb{N}_\infty$.

In order to relate the results proved inside type theory to the counterexample as given by the topological topos, let us first recall the definition of the latter and some its properties as relevant for this work. To build the topological topos, one starts with the monoid of continuous endomaps of the one-point compactification $\mathbb{N}_\infty$ of the discrete natural numbers, and then takes sheaves for the canonical topology of this monoid considered as a category. The category of sequential topological spaces is fully embedded into the topological topos. Their subobjects are precisely the Kuratowski limit spaces (sets equipped with convergent sequences subject to suitable axioms), which form a locally cartesian closed subcategory of the topological topos, and an exponential ideal. The image of the Yoneda embedding is $\mathbb{N}_\infty$. It is well-known, and easy to check, that in any topological space, and also in any limit space $X$, the convergent sequences are precisely the continuous maps $\mathbb{N}_\infty \to X$. Although the topological topos hosts the sequential topological spaces and the Kuratowki limit spaces, many of its objects are not (limit or topological) spaces, including the subobject classifier and the interpretation of the type-theoretic universe following [15]. However, the limit spaces and the sequential topological spaces form two different full reflective subcategories of the topological topos.

When the above theorem in type theory is interpreted in the topological topos, it gives that the quotient of $U$ by $\simeq$ is indiscrete. But, as discussed above, $U$ itself is indiscrete, and this can be formulated by saying that all maps into the Sierpiński space (open-subobject classifier) are constant. This also gives that all maps from $U$ to the two-point discrete space are constant, which is a form of Rice's Theorem for the universe, saying that all decidable predicates on $U$ are trivial. We also formulate and prove this internally in type theory: if a non-trivial, extensional, decidable property exists, then WLPO holds.

Notice that the simplicial sets model of univalence does validate WLPO since it actually validates classical logic. It remains as an open question whether univalence is consistent with continuity principles that entail the negation of WLPO.

In Section 2, we prove in type theory that the universe is indiscrete in the above sense, and then derive a version of Rice's Theorem for the universe from this. In Section 3, we look at this from a semantical point of view. In realizability models and in the topos of reflexive graphs, the functions from the universe to the booleans need not be extensional, in principle, but nevertheless they are all constant. The same phenomenon