Annals of Pure and Applied Logic 167 (2016) 806–819

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Positivity relations on a locale

Francesco Ciraulo [a,*], Steven Vickers [b]

[a] Department of Mathematics, University of Padua, Via Trieste 63, 35121 Padova, Italy
[b] School of Computer Science, University of Birmingham, Birmingham, B14 2TT, United Kingdom

ARTICLE INFO

Article history:
Available online 20 April 2016

MSC:
primary 03F55
secondary 54H12, 06B23, 06D22

Keywords:
Formal topology
Basic picture
Powerlocale

ABSTRACT

This paper analyses the notion of a *positivity relation* of Formal Topology from the point of view of the theory of Locales. It is shown that a positivity relation on a locale corresponds to a suitable class of points of its lower powerlocale. In particular, closed subtopologies associated to the positivity relation correspond to overt (that is, with open domain) weakly closed sublocales. Finally, some connection is revealed between positivity relations and localic suplattices (these are algebras for the powerlocale monad).

© 2016 Elsevier B.V. All rights reserved.

0. Introduction

Much of the theory of locales can be developed in a fully predicative way provided that "bases" are assumed as given data. Of course, this makes no difference within an impredicative setting where any locale has a base. Also predicatively, however, requiring bases does not appear as a real restriction, for there seems to be no other way to define a locale but presenting it by generators (hence at least a subbase) and relations (in a suitable sense). In Formal Topology (that is, predicative pointfree Topology) a presentation of a locale usually takes the form of a "cover relation" on a set. In [7] the name *formal topology* was given to a cover relation with a unary positivity predicate: this corresponds to the case of overt (or open) locales. In [9] a new definition, called *positive topology*, is proposed in which a binary relation replaces the positivity predicate. This positivity relation is used to define *formal closed subsets*, which give a suitable notion of *closed* subtopologies.

The main aim of this paper is to characterize positivity relations in a base-independent way (at the cost of introducing some impredicativity). In other words, we find the unknown value x in the proportion: formal topology is to overt locale as positive topology is to x.

We show that each formal closed subset is "splitting" (it has inhabited intersections with all covers of its elements) and that a positivity relation corresponds to a sub-suplattice of the suplattice of all splitting

* Corresponding author.
E-mail addresses: ciraulo@math.unipd.it (F. Ciraulo), s.j.vickers@cs.bham.ac.uk (S. Vickers).

http://dx.doi.org/10.1016/j.apal.2016.04.009
0168-0072/© 2016 Elsevier B.V. All rights reserved.

subsets. This leads to a number of characterizations of what a positivity relation on a locale is. In particular, it follows that each formal closed subset is a point of the lower powerlocale and thus corresponds to an overt weakly closed sublocale.

Further, we show some connections between positivity relations and *localic suplattices*, as introduced in [6], which are algebras for the powerlocale monad. Classes of points of localic suplattices give rise to positivity relations (and vice versa with classical logic).

A positivity relation on a locale $\mathcal{L}$ can also be read as a condition for selecting a class of points of $\mathcal{L}$. This idea becomes particularly clear when the positivity arises from localic sub-suplattice of the lower powerlocale $\mathcal{P}_L\mathcal{L}$.

To make the paper as general as possible, we begin from positivity relations on suplattices (with reversed morphisms). This is essentially the category of *basic topologies* [9].

1. Suplattices

We start by summarizing some of the impredicative facts about suplattices (complete join semilattices) and suplattice homomorphisms (join-preserving maps). Most of these are well known.

If L and M are suplattices, then so is $\mathbf{SupLat}(L, M)$, the set of suplattice homomorphisms $L \to M$, with joins calculated argumentwise: $\varphi \leq \psi$ when $\varphi(x) \leq \psi(y)$ for all $x \in L$.

For $1 = \{*\}$, let $\Omega \overset{def}{=} \mathsf{Pow}(1)$, the powerset of 1. In topos theory, this is the subobject classifier. Ω is the free suplattice over $\{*\}$, with injection of generators $* \mapsto \{*\}$ – in fact, for any set I, the powerset $\mathsf{Pow}(I)$ is the free suplattice over I. It follows that elements of a suplattice L are equivalent to suplattice homomorphisms $\Omega \to L$.

Because a suplattice L also has all meets – though we do not require homomorphisms to preserve them – it follows that L^{op} is also a suplattice. Moreover, a suplattice homomorphism $f: L \to M$ has a right adjoint g, which preserves all meets and hence is a suplattice homomorphism $M^{op} \to L^{op}$. This provides a self-duality $L \leftrightarrow L^{op}$ on the category of suplattices. It follows that elements of L^{op}, equivalent to suplattice homomorphisms $\Omega \to L^{op}$, are also equivalent to suplattice homomorphisms $L \to \Omega^{op}$.

However, we shall be particularly interested in suplattice homomorphisms $L \to \Omega$. *Classically,* with $\Omega^{op} \cong \Omega$, these are again equivalent to elements of L^{op}. More generally they are different.

We obtain two functors $\mathbf{SupLat}(-, \Omega), \Omega^{-}: \mathbf{SupLat}^{op} \to \mathbf{SupLat}$ acting on morphisms by composition, and with a natural transformation from the first to the second.

Since arbitrary maps $\varphi: L \to \Omega$ are equivalent to subsets $\varphi^{-1}(1)$ of L, we should identify which subsets correspond to the suplattice homomorphisms.

Definition 1.1. Let L be a suplattice. A subset $Z \subseteq L$ is *splitting* if

$$Z \ni x \leq \bigvee Y \Longrightarrow Z \between Y \text{ for every } x \in L \text{ and every } Y \subseteq L,$$

where, following Sambin, by $X \between Y$ we mean that $X \cap Y$ is inhabited. We write $\mathsf{Split}(L)$ for the collection of all splitting subsets of L.

Splitting subsets can be characterized also by the following two conditions

1. if $x \in Z$ and $x \leq y$, then $y \in Z$ (upward closed)
2. if $(\bigvee Y) \in Z$, then $y \in Z$ for some $y \in Y$ (completely prime)

and so they can as well be called *completely-prime upsets*. More succinctly, they can be characterized by a single condition that $(\bigvee Y) \in Z$ if and only if $Y \between Z$ – the "if" direction gives the upward closedness.

Download English Version:

https://daneshyari.com/en/article/4661568

Download Persian Version:

https://daneshyari.com/article/4661568

Daneshyari.com