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Priestley-type dualities for partially ordered structures

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A B S T R A C T

We introduce a general framework for generating dualities between categories of partial orders and categories of ordered Stone spaces; we recover in particular the classical Priestley duality for distributive lattices and establish several other dualities for different kinds of partially ordered structures.

1. Introduction

In this paper we give a topos-theoretic interpretation of Priestley duality for distributive lattices (cf. [15] and [16]), leading to natural analogues of this duality for other categories of partially ordered structures. Specifically, we establish dualities between various categories of ordered structures and categories of Priestley spaces which can be intrinsically characterized through appropriate separation axioms analogously to the case of the classical duality.

In order to build these 'Priestley-type' dualities, we investigate the classical duality from both a topological and an algebraic viewpoint. As it is well-known, topologically the duality is based on the patch topology construction, while algebraically the Boolean algebra of clopen sets of the Priestley space associated to a distributive lattice can be characterized as the free Boolean algebra on it. The unification between the

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algebraic and the topological formulations of the duality is conveniently provided by the notion of topos; in fact, the toposes involved in Priestley-type dualities admit, on one hand, an algebraic representation (as categories of sheaves on a preordered structure with respect to an appropriate Grothendieck topology on it) and on the other hand a topological one (as categories of sheaves on suitable spectra of these structures, as provided by the techniques of [4]).

Topologically, our 'Priestley-type' dualities are built by considering natural spectra for the given partially ordered structures, generating patch-type topologies from them and equipping the resulting spaces with the specialization orderings on the original spectra; algebraically, the dualities are obtained by assigning to any given ordered structure a Boolean algebra which is free on it (in an appropriate sense), equipped with a natural ordering on the points of its spectrum. Specific examples of dualities generated through this method are given in Section 5, and notably include 'Priestley-type' dualities for coherent posets, meet-semilattices and disjunctively distributive lattices.

We also argue more generally that various kinds of free structures provide a natural way for building dualities, to the extent that many free functors admit an inverse defined on an appropriate subcategory. Further illustrations of this phenomenon are provided in Section 5.

The structure of the paper is as follows. In Section 2 we carry out a general analysis of free structures and their construction via syntactic categories, with a particular emphasis on the construction of free Boolean algebras on different kinds of preordered structures through Morleyizations. We address in particular the problem of realizing free structures topologically and establish several results which allow us to identify, under natural hypotheses, a free structure on a poset as a structure generated by it inside an appropriate powerset. In Section 3 we present our topos-theoretic interpretation of Priestley duality, leading to the general method for building 'Priestley-type' dualities described in Section 4. We conclude with a section devoted to concrete examples of dualities generated through our methodology.

2. Free structures and Morleyizations

Let us start with some general remarks about the relationship between free structures and (generalized) syntactic categories.

2.1. Free structures and syntactic categories

As remarked in [4], the theory of syntactic categories can be profitably applied to the problem of constructing structures presented by generators and relations. In fact, any syntactic category of a given theory $\mathbb{T}$ can be regarded, in a sense that we shall not make precise in the present paper, as a structure presented by a set of 'generators', given by the sorts in the signature of the theory $\mathbb{T}$, subject to 'relations' expressed by the axioms of the theory $\mathbb{T}$. Conversely, to any structure $\mathcal{C}$ one can attach a *canonical signature* $\Sigma_{\mathcal{C}}$ to express 'relations' holding in $\mathcal{C}$, consisting of one sort $\ulcorner c \urcorner$ for each element c of $\mathcal{C}$ and possibly function or relation symbols whose canonical interpretation in $\mathcal{C}$ coincide with specified functions or subsets in $\mathcal{C}$ in terms of which the designated 'relations' holding in $\mathcal{C}$ can be formally expressed; over such a canonical signature one can then write down axioms possibly involving generalized connectives and quantifiers so to obtain a $\mathcal{S}$-theory (in the sense of Section 8 of [3]) $\mathbb{T}$ whose $\mathcal{S}$-syntactic category $\mathcal{C}_{\mathbb{T}}^{\mathcal{S}}$ can be identified with 'the free structure on $\mathcal{C}$ subject to the relations R', meaning that the $\mathcal{S}$-structure $\mathcal{D}$ in which the relations R are satisfied naturally correspond to the $\mathcal{S}$-homomorphism $\mathcal{C}_{\mathbb{T}}^{\mathcal{S}} \to \mathcal{D}$, in a way which can be concretely described as follows. To any $\mathcal{S}$-structure $\mathcal{D}$ we can canonically associate a $\mathcal{S}$-homomorphism $\mathcal{C} \to \mathcal{D}$, assigning to any element c of $\mathcal{C}$ the interpretation of $\ulcorner c \urcorner$ in $\mathcal{D}$; in particular we have a canonical $\mathcal{S}$-morphism $i : \mathcal{C} \to \mathcal{C}_{\mathbb{T}}^{\mathcal{S}}$, in terms of which the universal property of $\mathcal{C}_{\mathbb{T}}^{\mathcal{S}}$ can be expressed by saying that any $\mathcal{S}$-homomorphism $f : \mathcal{C} \to \mathcal{D}$ to a $\mathcal{S}$-structure $\mathcal{D}$ in which the relations R are satisfied can be extended, uniquely up to isomorphism, along the canonical morphism i, to a $\mathcal{S}$-homomorphism $\mathcal{C}_{\mathbb{T}}^{\mathcal{S}} \to \mathcal{D}$.