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Superstability and symmetry

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ABSTRACT

This paper continues the study of superstability in abstract elementary classes (AECs) satisfying the amalgamation property. In particular, we consider the definition of μ-superstability which is based on the local character characterization of superstability from first order logic. Not only is μ-superstability a potential dividing line in the classification theory for AECs, but it is also a tool in proving instances of Shelah's Categoricity Conjecture.

In this paper, we introduce a formulation, involving towers, of symmetry over limit models for μ-superstable abstract elementary classes. We use this formulation to gain insight into the problem of the uniqueness of limit models for categorical AECs.

1. Introduction

Finding an independence relation for abstract elementary classes (AECs) that satisfies symmetry is a long-standing problem. Oftentimes symmetry is just assumed as an axiom (e.g. Shelah's good frames [11]). Up until now, when it is derived, it is usually under additional model-theoretic assumptions such as tameness and/or set-theoretic assumptions such as or the existence of large cardinals. Some examples of the contexts and assumptions are tameness and the extra assumption of the extension property [1]; tameness and superstability assumptions above a sufficiently large cardinal [19]; $2^\lambda < 2^{\lambda^+}$, weak GCH, and categoricity in several successive cardinalities [10]; and $L_{\kappa,\omega}$-theories where κ a strongly compact cardinal [8]. In all of these examples symmetry is developed through a forking calculus. We provide a mechanism for deriving symmetry in abstract elementary classes without having to assume set-theoretic assumptions or tameness,

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and our methods of using towers to derive symmetry differ from these earlier results. Additionally, our point of view is localized as we consider only superstability and symmetry in μ and μ^+.

In this paper we identify a necessary and sufficient condition for symmetry of non-μ-splitting over limit models (see Definition 3) that involves reduced towers (Theorem 3). This new condition is particularly interesting since it does not have a pre-established first-order analog.

A preliminary test for our definition of symmetry is to derive it from the assumption of categoricity:

Corollary 1. *Suppose that $\mathcal{K}$ satisfies the amalgamation and joint embedding properties and μ is a cardinal $\geq \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$. If $\mathcal{K}$ is categorical in $\lambda = \mu^+$, then $\mathcal{K}$ has symmetry for non-μ-splitting over limit models.*

This examination of symmetry occurs in the context of superstable abstract elementary classes. The sharp dividing line for superstability in the classification theory of AECs is still uncertain in general. Grossberg and Vasey discuss the different shades of superstability and prove that under tameness they are all equivalent [6]. This paper helps to illuminate the dividing line in classes that may fail to be tame by considering the local character characterization of μ-superstability (see Assumption 1). In particular, we extend the work of [13–15], and [5] to derive the uniqueness of limit models of cardinality μ in μ-superstable AECs which satisfy μ-symmetry over limit models.

The series of papers [13–15], and [5] aims to verify the still-open conjecture that the uniqueness of limit models of cardinality μ is equivalent to superstability. Grossberg and Boney take a different approach to the uniqueness of limit models under the additional assumptions of tameness and a symmetry condition [1]. We refer the reader to [5] and [14] for a literature review on the uniqueness of limit models in abstract elementary classes. Briefly, the uniqueness of limit models has been used as a step to prove instances of Shelah's Categoricity Conjecture (e.g. [9,3,4]), as a mechanism for finding (Galois-)saturated models in singular cardinals [9], as a tool to derive the amalgamation property [7], and as a lens to examine strictly stable AECs [2].

The organization of the proofs in the series [13–15], and [5] is to transfer some model-theoretic structure from assumptions in a higher cardinality down to μ. Here, in Theorem 1, we extend this line of reasoning by weakening the structural assumptions on $\mathcal{K}_{\mu^+}$.

Theorem 1. *Let $\mathcal{K}$ be a μ-stable abstract elementary class satisfying the amalgamation and joint embedding properties. Suppose $\mathcal{K}$ satisfies the locality and continuity properties of μ-splitting (see Assumption 1) and also satisfies the property that any union of saturated models of cardinality μ^+ is saturated. For $N \in \mathcal{K}_\mu$ and limit ordinals, $\theta_1, \theta_2 < \mu^+$, if M_l is (μ, θ_l)-limit model for $l \in \{1, 2\}$, then M_1 and M_2 are isomorphic over N.*

More interestingly, the proof of Theorem 1 leads to the isolation of the question of whether or not every reduced tower is continuous. Here we see that this is equivalent to asking whether or not the abstract elementary class has a non-μ-splitting relation that satisfies symmetry over limit models. This provides some insight into the strength of the assumptions that might be required to prove the uniqueness of limit models.

Later research has established that our formulation of symmetry is equivalent to other existing forms and that symmetry for non-μ-splitting over limit models follows from μ-superstability in tame AECs [18]. Our formulation of symmetry has also been used to derive the following results: μ-superstability implies the uniqueness of limit models in tame AECs [18, Corollary 1.4]; the union of an increasing chain of μ^+-saturated models is μ^+-saturated in μ- and μ^+-superstable classes that satisfy symmetry for non-μ^+-splitting over limit models [16, Theorem 1]; and some level of uniqueness of limit models can be recovered in strictly μ-stable AECs which satisfy a weakening of Definition 3 [2, Theorem 1]. Moreover, Vasey has used our results to lower the bound of Shelah's Downward Categoricity Transfer Theorem [21, Corollary 7.12].