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Infinite games specified by 2-tape automata

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ABSTRACT

We prove that the determinacy of Gale–Stewart games whose winning sets are infinitary rational relations accepted by 2-tape Büchi automata is equivalent to the determinacy of (effective) analytic Gale–Stewart games which is known to be a large cardinal assumption. Then we prove that winning strategies, when they exist, can be very complex, i.e. highly non-effective, in these games. We prove the same results for Gale–Stewart games with winning sets accepted by real-time 1-counter Büchi automata, then extending previous results obtained about these games.

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that: (a) there is a model of ZFC in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$ but σ cannot be recursive and not even in the class $(\Sigma_2^1 \cup \Pi_2^1)$; (b) there is a model of ZFC in which the game $G(L(\mathcal{A}))$ is not determined.
2. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that $L(\mathcal{A})$ is an arithmetical Δ_3^0-set and Player 2 has a winning strategy in the game $G(L(\mathcal{A}))$ but has no hyperarithmetical winning strategies in this game.
3. There exists a recursive sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $\mathcal{A}_n$, $n \geq 1$, such that all games $G(L(\mathcal{A}_n))$ are determined, but for which it is Π_2^1-complete hence highly undecidable to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$.

Then we consider the strengths of determinacy for these games, and we prove the following results.

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}_\sharp$ such that the game $G(\mathcal{A}_\sharp)$ is determined iff the effective analytic determinacy holds.
2. There is a transfinite sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, such that the games $G(L(\mathcal{A}_\alpha))$ have strictly increasing strengths of determinacy.

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We also show that the determinacy of Wadge games between two players in charge of infinitary rational relations accepted by 2-tape Büchi automata is equivalent to the (effective) analytic Wadge determinacy and thus also equivalent to the (effective) analytic determinacy.

1. Introduction

In Computer Science, non-terminating systems in relation with an environment may be specified with some particular infinite games of perfect information, called Gale Stewart games since they have been firstly studied by Gale and Stewart in 1953 in [14]. The two players in such a game are respectively a non-terminating reactive program and the "environment". A Gale–Stewart game is defined as follows. If X is a (countable) alphabet having at least two letters and $A \subseteq X^\omega$, then the Gale–Stewart game $G(A)$ is an infinite game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on .... After ω steps, the two players have composed an infinite word $x = a_1 b_1 a_2 b_2 \dots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play. The game $G(A)$ is said to be determined iff one of the two players has a winning strategy.

Then the problem of the synthesis of winning strategies is of great practical interest for the problem of program synthesis in reactive systems. In particular, if $A \subseteq X^\omega$, where X is here a finite alphabet, and A is effectively presented, i.e. accepted by a given finite machine or defined by a given logical formula, the following questions naturally arise, see [35,20]: (1) Is the game $G(A)$ determined? (2) If Player 1 has a winning strategy, is it effective, i.e. computable? (3) What are the amounts of space and time necessary to compute such a winning strategy? Büchi and Landweber gave a solution to the famous Church's Problem, posed in 1957, by proving that in a Gale Stewart game $G(A)$, where A is a regular ω-language, one can decide who the winner is and compute a winning strategy given by a finite state transducer, see [36]. Walukiewicz extended Büchi and Landweber's Theorem to the case of a winning set A which is deterministic context-free, i.e. accepted by some deterministic pushdown automaton, answering a question of Thomas and Lescow in [35,20]. He first showed in [38] that one can effectively construct winning strategies in parity games played on pushdown graphs and that these strategies can be computed by pushdown transducers. Notice that later some extensions to the case of higher-order pushdown automata have been established [3,4].

In [12,13] we have studied Gale–Stewart games $G(A)$, where A is a context-free ω-language accepted by a *non-deterministic* pushdown automaton, or even by a 1-counter automaton. We have proved that the determinacy of Gale–Stewart games $G(A)$, whose winning sets A are accepted by real-time 1-counter Büchi automata, is equivalent to the determinacy of (effective) analytic Gale–Stewart games. On the other hand Gale–Stewart games have been much studied in Set Theory and in Descriptive Set Theory, see [18,17,22,29]. It has been proved by Martin that every Gale–Stewart game $G(A)$, where A is a Borel set, is determined [18]. Notice that this is proved in ZFC, the commonly accepted axiomatic framework for Set Theory in which all usual mathematics can be developed. But the determinacy of Gale–Stewart games $G(A)$, where A is an (effective) analytic set, is not provable in ZFC; Martin and Harrington have proved that it is a large cardinal assumption equivalent to the existence of a particular real, called the real $0^\sharp$, see [17, p. 637]. Thus we proved in [12,13] that the determinacy of Gale–Stewart games $G(A)$, whose winning sets A are accepted by real-time 1-counter Büchi automata, is also equivalent to the existence of the real $0^\sharp$, and thus not provable in ZFC.

In this paper we consider Gale–Stewart games $G(L(\mathcal{A}))$, where $L(\mathcal{A})$ is an infinitary rational relation, i.e. an ω-language over a product alphabet $X = \Sigma \times \Gamma$, which is accepted by a 2-tape (non-deterministic) Büchi automaton $\mathcal{A}$. In such a game, the two players alternatively write letters from the product alphabet $X = \Sigma \times \Gamma$, and after ω steps they have produced an infinite word over X which may be identified with