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Reverse mathematical bounds for the Termination Theorem

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A B S T R A C T

In 2004 Podelski and Rybalchenko expressed the termination of transition-based programs as a property of well-founded relations. The classical proof by Podelski and Rybalchenko requires Ramsey's Theorem for pairs which is a purely classical result, therefore extracting bounds from the original proof is non-trivial task. Our goal is to investigate the termination analysis from the point of view of Reverse Mathematics. By studying the strength of Podelski and Rybalchenko's Termination Theorem we can extract some information about termination bounds.

1. Introduction

In [34] Podelski and Rybalchenko characterized the termination of transition-based programs as a property of well-founded relations. Their result may be stated as follows: a binary relation R is well-founded if and only if there exist a natural number k and k-many well-founded relations whose union contains the transitive closure of R. The classical proof of Podelski and Rybalchenko's Termination Theorem (just Termination Theorem for short) requires Ramsey's Theorem for pairs. Although Ramsey's Theorem for pairs is a purely classical result, the Termination Theorem can be intuitionistically proved by using some intuitionistic version of Ramsey and providing to consider the intuitionistic notion of well-foundedness. In 2012 Vytiniotis, Coquand and Wahlstedt proved an intuitionistic version of the Termination Theorem by using the Almost-Full Theorem [38], while in 2014 Stefano Berardi and the first author proved it by using

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the H-closure Theorem [6]. The H-closure Theorem arose by the combinatorial fragment needed to prove the Termination Theorem (see Sections 2, 3).

The goal of this paper is to study the H-closure Theorem and the Termination Theorem from the viewpoint of Reverse Mathematics, in order to extract bounds for termination. The first question is whether the H-closure Theorem and the Termination Theorem are equivalent over RCA_0 to Ramsey's Theorem for pairs. Due to our analysis we answer to [14, Open Problem 2] posed by Gasarch: finding a natural example showing that the Termination Theorem requires the full Ramsey Theorem for pairs. In this paper we prove that such program cannot exists. We also answer negatively to [14, Open Problem 3] posed by Gasarch: is the Termination Theorem equivalent to Ramsey's Theorem for pairs?

In [13] Figueira et al. gave a deeper analysis of the Termination Theorem by using Dickson's Lemma.[2] In fact the Termination Theorem is a consequence of Dickson's Lemma by observing that any relation is well-founded if and only if it is embedded into a well-quasi-ordering. However this property of well-quasi-orderings is equivalent to ACA_0 over RCA_0 and therefore in order to analyze the strength of the Termination Theorem we need a different point of view.

In Section 4 we prove that the Termination Theorem is equivalent over RCA_0 to a weak version of Ramsey's Theorem for pairs. As a corollary of this result we have that for any natural number k, CAC (the Chain–AntiChain principle) is stronger than the Termination Theorem for k many relations, which is the statement: given a relation R, if there exist k-many well-founded relations $R_0, \dots, R_{k-1}$ such that $R_0 \cup \dots \cup R_{k-1} \supseteq R^+$ then R is well-founded. Therefore we get answers to [14, Open Problem 2, Open Problem 3].

These results can be used to characterize the programs proved to be terminating by the Termination Theorem: our goal is to extract a time bound for such a program by using reverse mathematics tools. Assume that R is the binary transition relation of some program. We say that a function $f : S \to \mathbb{N}$ is a bound for the relation R on S, if any R-decreasing sequence starting from an element $a \in S$ is shorter than $f(a)$. By using [12,31,10] it is known that the class of provably recursive functions of $\mathsf{WKL}_0 + \mathrm{CAC}$ is exactly the same as the class of primitive recursive functions. Hence given any binary relation R generated by a primitive recursive transition function, we conclude that if there exist k-many relations $R_1 \cup \dots \cup R_{k-1} \supseteq R^+$ with primitive recursive bounds, then the program has a primitive recursive bound. The proof is in Section 6.

In order to provide more precise termination bounds, in Section 7 we study the reverse mathematical strength of some bounded versions of both the H-closure Theorem and of the Termination Theorem. Differently from the full case, in the restricted ones they turn out to be equivalent. Moreover we prove they are equivalent to a weaker version of the Paris Harrington Theorem [30].

A natural question which arise is: is there a correspondence between the complexity of a primitive recursive transition relation and the number of relations which compose the transition invariant? Due to our analysis and by using the relationship between Paris Harrington Theorem and the Fast-Growing Hierarchy in Section and 8 we provide results in this directions.

Finally, in Section 9, we focus on the case of iterated applications of the Termination Theorem.

2. Ramsey's Theorem in reverse mathematics

Ramsey's Theorem, and in particular Ramsey's Theorem for pairs in two colors, is a central argument of study in Reverse Mathematics. In this section we summarize some main facts about the strength of Ramsey's Theorem and of some of its corollaries we will use in this paper. Let k be a natural number. Ramsey's Theorem for pairs in k colors guarantees that for any coloring in k-many colors over the edges of

[2] Dickson's Lemma states that $(\mathbb{N}^k, \leq)$ (where $\leq$ is the componentwise order) is a well-quasi order [22,27]; i.e. every infinite sequence σ of elements of $\mathbb{N}^k$ is such that there exists $n < m$ with $\sigma(n) \leq \sigma(m)$.