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Inductive inference and reverse mathematics

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ABSTRACT

The present work investigates inductive inference from the perspective of reverse mathematics. Reverse mathematics is a framework that allows gauging the proof strength of theorems and axioms in many areas of mathematics. The present work applies its methods to basic notions of algorithmic learning theory such as Angluin's tell-tale criterion and its variants for learning in the limit and for conservative learning, as well as to the more general scenario of partial learning. These notions are studied in the reverse mathematics context for uniformly and weakly represented families of languages. The results are stated in terms of axioms referring to induction strength and to domination of weakly represented families of functions.

1. Introduction

It is standard practice in mathematics to use known theorems to prove others. In these cases it can often be observed that some theorem T seems to be "stronger" than another theorem U in the sense that T allows proving U, but not vice versa. In the 1970s, Friedman [12] proposed a framework that formalises this intuition and allows gauging the different strength levels of theorems found in classical mathematics. The general idea is to assume only a subset of the axioms of second order arithmetic which by itself is too

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weak to prove the theorems in question, and then to analyse whether one theorem implies the other over this weak base system. Of course, if we want to *exactly* determine the strength of a mathematical theorem T in this sense, then we need to look at both directions: which theorems are implied by T and which imply T? As all of mathematics is ultimately founded on axioms, it is a natural next step to extend this study to the relation between axioms and theorems, and to wonder what *axioms* are exactly equivalent to a given theorem T, that is, imply T and are implied by T.

This "inverted" approach — where one uses theorems to prove axioms instead of the other way around — explains the name of this field of study: reverse mathematics. The subject has developed well since its inception, in particular thanks to many substantial contributions made by Simpson and his students [24].

The methodology of reverse mathematics has been applied to many fields of classical mathematics, for example to group theory, to vector algebra, to analysis and — especially in recent years — to combinatorics, including Ramsey theory and related fields. We refer to the books of Hirschfeldt [15] and Simpson [24] which are convenient resources for the topic and give many references.

In this article we propose to apply the methodology of reverse mathematics to the field of inductive inference, or algorithmic learning theory, as introduced and studied in numerous publications during the last decades [1,2,4,9,5,13,18,23,27]. A main focus of our work will be Angluin's tell-tale criterion for learnability of families of sets [2], one of the central results in classical algorithmic learning theory. We will also study several of its variants and a number of related results. We would like to point out that reverse mathematics analyses of inductive inference have also been proposed by de Brecht and Yamamoto [10] and by Hayashi [14] but that their approaches to the topic are unrelated to ours and to each other.

As algorithmic learning theory in its classical form is usually about the learnability of a language drawn from a family of languages that was fixed in advance, using a learner constructed specifically for that family, it is important for our study to know which such families are even guaranteed to exist over a weak base system as used in reverse mathematics. To handle this complication we define two ways of representing such a family by a single subset of the natural numbers: uniformly and weakly represented families of sets. Then when the set representing a family exists in the weak base system, so does the family. Similarly we will define weakly represented families of *functions* which will be used to formulate one of the reverse mathematics axioms used here.

As we will show in this article, many results in inductive inference relate to the following three axioms from reverse mathematics, which we only state informally for the time being.

- The axiom DOM which states that for every weakly represented family of functions there exists a function growing faster than all members of the family;
- The axiom ACA_0 which states that the Turing jump of every set exists;
- The axiom $\mathrm{I}\Sigma_2$ that allows inductive arguments on sets definable via Σ_2-formulas.

The remainder of this paper is organised as follows: In Section 2 we will introduce important definitions from reverse mathematics and inductive inference. Section 3 will discuss what bearing the tell-tale criterion has on the learnability of weakly represented families. The axiom DOM will be identified as necessary and sufficient for this. In Section 4 we will then follow Angluin's approach more closely and investigate the learnability of uniformly represented families, the closest equivalent in reverse mathematics of the indexed families that Angluin originally studied. We will show that for the learnability of uniformly represented families the degree of effectiveness of the bound in the tell-tale criterion is crucial. In this context we will also study conservative learning, where the axiom ACA_0 will be of relevance. Section 5 then focuses on sufficient criteria for learning from the classical theory and shows that they are sufficient for the learnability of uniformly represented families in reverse mathematics as well. However, for weakly represented families we will again require the additional axiom DOM. In Section 6 we will study the situation for partial learning where the axiom $\mathrm{I}\Sigma_2$ will play an important role. We point out that a preliminary version of this article