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Classifying torsion free groups in o-minimal expansions of real closed fields

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ABSTRACT

In this paper we classify modulo definable group isomorphisms all torsion free groups of dimension up to four definable in an o-minimal expansion of a real closed field satisfying some conjectural assumptions. This classification implies that the number of uniformly definable families of torsion free definable groups of dimension up to four is finite and depends only on whether or not an exponential function is definable in the structure.

We also adapted the Lyndon–Hochschild–Serre spectral sequence approach to the category of groups and modules definable in a fixed o-minimal structure to compute the second cohomology groups (in the o-minimal group cohomology) in the particular cases that were needed. This provides a blueprint of how to move beyond dimension four once the solvable real Lie groups are classified in higher dimensions.

1. Introduction

Throughout this introduction we work inside an o-minimal expansion $\mathcal{R}$ of a real closed field, and by definable we mean $\mathcal{R}$-definable.

It is known that for every definable group G there is a definable group extension $1 \to R \to G \to G/R \to 1$ where R is the solvable radical of G (the maximal solvable definably connected definable normal subgroup of G) [6]. Then, the definable quotient G/R is either definably semisimple or finite. Now, by Proposition 2.2 in [4], we know that if H is the maximal normal torsion free definable subgroup of R, then R/H is definably isomorphic to a definably compact (abelian) definable group C. Thus, in order to understand the possible definable groups in $\mathcal{R}$, one needs to first understand the possible definably semisimple or finite groups

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G/R, the possible definably compact groups C, and the torsion free groups H, then understand the possible actions between them, the definable group extensions that can arise, and finally classify them up to definable group isomorphisms.

Groups definable in $\mathcal{R}$ have been popular subjects of study for the last decades, and some amazing results have been achieved for both definably semisimple and definably compact definable groups (see [14] for a good survey on the field). Definably semisimple definable groups are classified in [15, Theorem 4.1]. In that paper Peterzil, Pillay, and Starchenko proved that every definably connected semisimple centerless definable group is definably isomorphic to a linear semialgebraic group.

The semialgebraic groups in an o-minimal expansion of a real closed field depend on the ground structure (an elliptic curve with infinitesimal j-invariant is not definably isomorphic to any of the standard elliptic curves), but the above result implies that the definably connected definably semisimple groups definable in $\mathcal{R}$ do not depend on the language we are using: the language of ordered fields is enough to define all possible definably connected definably semisimple definable groups in the structure (modulo definable isomorphisms).

For definably compact definable groups we have the following.

- Since we can add all possible analytic functions restricted to a compact domain to the real field and preserve o-minimality, every compact Lie group will be definable in an o-minimal expansion of $\mathbb{R}$.
- Given any definably compact group G definable in a saturated o-minimal expansion of a real closed field, one can take the quotient by its smallest type-definable subgroup of bounded index (which is normal and denoted G^{00}), and we get a compact Lie group (with the logic topology) whose dimension as Lie group is equal to the o-minimal dimension of G (see [10]).

This implies that definably compact definable groups depend heavily on the language of the structure. Also, it is clear that in a given theory the definably compact definable groups depend on the model (we can take lattices of the additive group of a real closed field by infinitesimal elements), but one can "recover" the corresponding compact Lie group over a saturated o-minimal expansion of a real closed field by taking the quotient by G^{00}.

Until recently, very little was known about the torsion free component of the decomposition. The second author, together with Annalisa Conversano and Sergei Starchenko, recently proved the following fact.

Fact 1.1. ([5]) Every torsion free group definable in an o-minimal theory is completely solvable.

Where by a completely solvable torsion free definable group we mean the following.

Definition 1.1. A torsion free definable group G is called *completely solvable* if there exist an increasing sequence of normal definable subgroups $G_0, G_1, \ldots, G_n$ of G such that $G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = \{0\}$ and G_{i+1}/G_i is a one dimensional torsion free group for all $i < n$.

In this paper, we begin a characterization of torsion free (completely solvable) groups definable in o-minimal expansions of a real closed field. It should be noted that a complete characterization is probably out of reach at the moment since we do not have a complete classification of completely solvable real Lie groups (a particular case of our aim, by results in [5]). In the real Lie group case, a workable classification of such groups has been obtained up to dimension four [1], and no list of such groups has been achieved in dimensions greater than seven.

The main result of this paper is a complete classification of the torsion free groups of dimensions less than five definable in an o-minimal expansion $\mathcal{R}$ of a real closed field assuming some conjectures about one and two dimensional definable groups, which are still unsolved and will be stated precisely in Section 2. This will be achieved under two scenarios: exponential and nonexponential o-minimal expansions of a real closed field (Sections 4, 5), results which will be summarized in Theorems 4.12 and 5.16.