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# The semantic isomorphism theorem in abstract algebraic logic

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ABSTRACT

One of the most interesting aspects of Blok and Pigozzi's algebraizability theory is that the notion of algebraizable logic $\mathcal{L}$ can be characterised by means of Syntactic and Semantic Isomorphism Theorems. While the Syntactic Isomorphism Theorem concerns the relation between the theories of the algebraizable logic $\mathcal{L}$ and those of the equational consequence relative to its equivalent algebraic semantics $\mathsf{K}$, the Semantic Isomorphism Theorem describes the interplay between the filters of $\mathcal{L}$ on an arbitrary algebra $\boldsymbol{A}$ and the congruences of $\boldsymbol{A}$ relative to $\mathsf{K}$. The pioneering insight of Blok and Jónsson, and the further generalizations by Galatos, Tsinakis, Gil-Férez and Russo, showed that the concept of algebraizability was not intrinsic to the connection between a logic and an equational consequence, thus inaugurating the abstract theory of equivalence between structural closure operators. However all these works focus only on the Syntactic Isomorphism Theorem, disregarding the semantic aspects present in the original theory. In this paper we fill this gap by introducing the notion of compositional lattice, which acts on a category of evaluational frames. In this new framework the non-linguistic flavour of the Semantic Isomorphism Theorem can be naturally recovered. In particular, we solve the problem of finding sufficient and necessary conditions for transferring a purely syntactic equivalence to the semantic level as in the Semantic Isomorphism Theorem.



## 1. Introduction

The theory of algebraizability was introduced by Blok and Pigozzi [5] as a common mathematical framework to describe the relations that hold between a logic and its algebraic semantics. In order to review its basic concepts, we recall that a class of algebras $\mathsf{K}$ is *generalized quasi-variety* if it can be axiomatized by a set of generalized quasi-equations, i.e. quasi-equations whose antecedent is a possibly infinite set of equations written with a *countable* set of variables. It is well known [4, Theorem 8.1] (see also [18]) that generalized quasi-varieties are exactly the classes of algebras closed under isomorphic copies, subalgebras, direct products and the class operator $\mathbb{U}$ defined for every class of algebras $\mathsf{K}$ as follows:

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$$\mathbb{U}(\mathsf{K}) := \{\boldsymbol{A} : \text{ the countably generated subalgebras of } \boldsymbol{A} \text{ belong to } \mathsf{K}\}.$$

Accordingly a logic $\mathcal{L}$ (formulated in a countable set of variables) is called *algebraizable* with respect to a generalised quasi-variety $\mathsf{K}$ if there are two structural transformers, of formulas into (sets of) equations and vice-versa, which allow to interpret the consequence of $\mathcal{L}$ into the equational consequence $\vDash_{\mathsf{K}}$ relative to $\mathsf{K}$ and which are one inverse to the other modulo the interderivability relation $=\!\mid\vDash_{\mathsf{K}}$. One of the central results in the theory asserts that a logic $\mathcal{L}$ is algebraizable with respect to the generalised quasi-variety $\mathsf{K}$ if and only if the lattices of *theories* of $\mathcal{L}$ and $\vDash_{\mathsf{K}}$ are isomorphic, once they are expanded with the actions of the monoid of *substitutions*. In abstract algebraic logic this characterisation of algebraizability is known as the Syntactic Isomorphism Theorem, since it makes reference only to the common structure shared by some linguistic objects, i.e. the theories of $\mathcal{L}$ and of those of $\vDash_{\mathsf{K}}$. But the strength of algebraizability comes also from the fact that the notion of syntactic equivalence, which is implicit in its definition, transfers to the semantic level. In fact, a logic $\mathcal{L}$ is algebraizable with respect to the generalised quasi-variety $\mathsf{K}$ if and only if for every algebra $\boldsymbol{A}$ the lattice of *filters* of $\mathcal{L}$ on $\boldsymbol{A}$ and the lattice of *congruences* relative to $\mathsf{K}$ are isomorphic, when expanded with the actions of the monoid of *endomorphisms* of $\boldsymbol{A}$. This result is known as the Semantic Isomorphism Theorem, since it moves the attention to the interplay between the models of $\mathcal{L}$ and of $\vDash_{\mathsf{K}}$. Precise statements of the two results will be given in Theorems 3.1 and 3.7 below.

As soon as it was recognized that algebraizability was a synonym of the special kind of deductive equivalence between $\mathcal{L}$ and $\vDash_{\mathsf{K}}$ expressed in the Syntactic Isomorphism Theorem, it became clear that a suitable generalisation of this notion would provide a framework to describe deductive equivalences between arbitrary structural closure operators, not necessarily defined on the formulas or equations of a given algebraic language. This intuition led Blok and Jónsson [4] to give the first abstract formulation of algebraizability within the context of $\mathcal{M}$-sets, where $\mathcal{M}$ is a monoid (whose elements play the role of the ordinary substitutions) acting on a set (which can be thought of as the set of formulas over which the consequence operator is defined, equipped with the corresponding substitutions). In this setting, emphasis shifted from the linguistic aspect of the equivalence (given by the existence of transformers from formulas into equations and vice versa) to its lattice-theoretic aspect, as expressed in the Syntactic Isomorphism Theorem. Accordingly, they defined two structural closure operators on different $\mathcal{M}$-sets to be equivalent if there is an isomorphism between the complete lattices of their closed sets expanded by the actions of $\mathcal{M}$, which induce unary operations on these lattices after closing under the corresponding closure operators.

In the last years the history of the abstract version of the Syntactic Isomorphism Theorem has gone very far. One of the first steps was done by Blok and Jónsson themselves, by proving that the "only if" part of the Syntactic Isomorphism Theorem holds in the context of $\mathcal{M}$-sets too. This could have lead to optimistic expectations, but unfortunately Gil-Férez provided a counterexample to its "if" part in [16] (see also [17]). Therefore two characterisations of algebraizability, which were equivalent in the original setting, turned out not to be so when moved to the more abstract context of $\mathcal{M}$-sets. From then on, the research focused on the problem of finding necessary and sufficient conditions under which the Syntactic Isomorphism Theorem could be recovered even in its abstract version.

The next step of the abstraction process was due to Galatos and Tsinakis in [15], which moved the study of algebraizability to the even more general context of modules over complete residuated lattices. Their idea is to split the equivalences in two halves, which they call structural representations. Then the quest became that of finding sufficient and necessary conditions (on a module over a complete residuated lattice) under which every structural representation is induced by a module morphism. The cornerstone of their work was the intuition that this problem could be elegantly solved with categorical tools: equipping the modules over a fixed complete residuated lattice with a categorical structure, they characterised the desired objects as