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Building independence relations in abstract elementary classes

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ABSTRACT

We study general methods to build forking-like notions in the framework of tame abstract elementary classes (AECs) with amalgamation. We show that whenever such classes are categorical in a high-enough cardinal, they admit a good frame: a forking-like notion for types of singleton elements.

Theorem 0.1 *(Superstability from categoricity). Let K be a $(< \kappa)$-tame AEC with amalgamation. If $\kappa = \beth_\kappa > LS(K)$ and K is categorical in a $\lambda > \kappa$, then:*

- *K is stable in any cardinal μ with $\mu \geq \kappa$.*
- *K is categorical in κ.*
- *There is a type-full good λ-frame with underlying class K_λ.*

Under more locality conditions, we prove that the frame extends to a global independence notion (for types of arbitrary length).

Theorem 0.2 *(A global independence notion from categoricity). Let K be a densely type-local, fully tame and type short AEC with amalgamation. If K is categorical in unboundedly many cardinals, then there exists $\lambda \geq LS(K)$ such that $K_{\geq \lambda}$ admits a global independence relation with the properties of forking in a superstable first-order theory.*

As an application, we deduce (modulo an unproven claim of Shelah) that Shelah's eventual categoricity conjecture for AECs (without assuming categoricity in a successor cardinal) follows from the weak generalized continuum hypothesis and a large cardinal axiom.

Corollary 0.3. *Assume $2^\lambda < 2^{\lambda^+}$ for all cardinals λ, as well as an unpublished claim of Shelah. If there exists a proper class of strongly compact cardinals, then any AEC categorical in* some *high-enough cardinal is categorical in* all *high-enough cardinals.*

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1. Introduction

Independence (or forking) is a central notion of model theory. In the first-order setup, it was introduced by Shelah [29] and is one of the main devices of his book. One can ask whether there is such a notion in the nonelementary context. In homogeneous model theory, this was investigated in [19] for the superstable case and [11] for the simple and stable cases. Some of their results were later generalized by Hyttinen and Kesälä [18] to tame and $\aleph_0$-stable finitary abstract elementary classes (AECs). For general[2] AECs, the answer is still a work in progress.

In [31, Remark 4.9.1] it was asked whether there is such a notion as forking in AECs. In his book on AECs [34], Shelah introduced the concept of good λ-frames (a local independence notion for types of singletons) and some conditions are given for their existence. Shelah's main construction (see [34, Theorem II.3.7]) uses model-theoretic and set-theoretic assumptions: categoricity in two successive cardinals and principles like the weak diamond.[3] It has been suggested[4] that replacing Shelah's strong local model-theoretic hypotheses by the global hypotheses of amalgamation and tameness (a locality property for types introduced by Grossberg and VanDieren [15]) should lead to better results with simpler proofs. Furthermore, one can argue that any "reasonable" AEC should be tame and have amalgamation, see for example the discussion in Section 5 of [6], and the introductions of [5] or [15]. In particular, they follow from a large cardinal axiom and categoricity:

Fact 1.1. Let K be an AEC and let $\kappa > \mathrm{LS}(K)$ be a strongly compact cardinal. Then:

(1) [5] K is $(< \kappa)$-tame (in fact fully $(< \kappa)$-tame and short).
(2) [26, Proposition 1.13][5] If $\lambda > \beth_{\kappa+1}$ is such that K is categorical in λ, then $K_{\geq \kappa}$ has amalgamation.

Examples of the use of tameness and amalgamation include [3] (an upward stability transfer), [25] (showing that tameness is equivalent to a natural topology on Galois types being Hausdorff), [16] (an upward categoricity transfer theorem, which can be combined with Fact 1.1 and the downward transfer of Shelah [31] to prove that Shelah's eventual categoricity conjecture for a successor follows from the existence of a proper class of strongly compact cardinals) and [4,9,20], showing that good frames behave well in tame classes.

Ref. [40] constructed good frames in ZFC using global model-theoretic hypotheses: tameness, amalgamation, and categoricity in a cardinal of high-enough cofinality. However we were unable to remove the assumption on the cofinality of the cardinal or to show that the frame was *ω-successful*, a key technical property of frames. Both in Shelah's book and in [40], the question of whether there exists a *global* independence notion (for longer types) was left open. In this paper, we continue working in ZFC with tameness and amalgamation, and make progress toward these problems. Regarding the cofinality of the categoricity cardinal, we show that it is possible to take the categoricity cardinal to be high-enough: (Theorem 10.16):

Theorem 10.16. *Let K be a $(< \kappa)$-tame AEC with amalgamation. If $\kappa = \beth_\kappa > LS(K)$ and K is categorical in a $\lambda > \kappa$, then there is a type-full good λ-frame with underlying class K_λ.*

As a consequence, the class K above has several superstable-like properties: for all $\mu \geq \lambda$, K is stable[6] in μ (this is also part of Theorem 10.16) and has a unique limit model of cardinality μ (by e.g. [9, Corollary 6.9]

[2] For a discussion of how the framework of tame AECs compare to other non first-order frameworks, see the introduction of [39].
[3] Shelah claims to construct a good frame in ZFC in [34, Theorem IV.4.10] but he has to change the class and still uses the weak diamond to show his frame is ω-successful.
[4] The program of using tameness and amalgamation to prove Shelah's results in ZFC is due to Rami Grossberg and dates back to at least [15], see the introduction there.
[5] This is stated there for the class of models of an $\mathbb{L}_{\kappa,\omega}$ theory but Boney [5] argues that the argument generalizes to any AEC K with $\mathrm{LS}(K) < \kappa$.
[6] The downward stability transfer from categoricity is an early result of Shelah [31, Claim 1.7], but the upward transfer is new and improves on [40, Theorem 7.5]. In fact, the proof here is new even when K is the class of models of a first-order theory.