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Abelian p-groups and the Halting problem

Rodney Downey, Alexander G. Melnikov, Keng Meng Ng

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ABSTRACT

We investigate which effectively presented abelian p-groups are isomorphic relative to the halting problem. The standard approach to this and similar questions uses the notion of Δ^0_2-categoricity (to be defined). We partially reduce the description of Δ^0_2-categorical p-groups of Ulm type 1 to the analogous problem for equivalence structures. Using this reduction, we solve a problem left open in [5]. For the sake of the reduction mentioned above, we introduce a new notion of effective Δ^0_2-categoricity that lies strictly in-between plain Δ^0_2-categoricity and relative Δ^0_2-categoricity (to be defined). We then reduce the problem of classifying effective Δ^0_2-categoricity to a question stated in terms of Σ^0_2-sets. Among other results, we show that for c.e. Turing degrees bounding such sets is equal to being complete.

1. Introduction

Following Mal'cev [20] and Rabin [22], we say that an algebraic structure is *computable* or *constructive* if there exists a numbering of its elements by natural numbers under which the operations, relations and equality become Turing computable. This numbering is called a computable presentation or constructivization of the structure. For example, a group has a computable presentation if and only if it has a "recursive presentation" (Higman [15]) with decidable word problem. This definition also generalizes the early notion of an "explicitly presented" field due to van der Waerden [25] (formally clarified by Fröhlich and Shepherdson [10]).

The general philosophy of effective algebra is that effectively presented objects should be studied under effective isomorphisms. Following the standard terminology [1,9], we say that a computable algebraic structure is *computably categorical* or *autostable* if every two computable presentations of the structure agree up to a computable isomorphism. Most non-trivial "natural" examples of computable algebraic structures are *not* computably categorical. For example, only very few abelian p-groups [24] are computably categorical, and those are trivial; see [14,9,1] for more examples. This paper contributes to a general framework (e.g., [2,21,3,7]) that investigates computable structures which are *not* computably categorical but are *close* to being computably categorical (to be explained).

E-mail address: alexander.g.melnikov@gmail.com (A.G. Melnikov).

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In contrast to computably categorical structures that are rare, computable structures that are isomorphic *relative to* the halting problem $0'$, or maybe relative to a few iterations of the halting problem, often occur in mathematical practice.[1] That is, if we had an oracle for $0^{(n)}$, we could compute an isomorphism. Intuitively, it means that to build an isomorphism it is sufficient to understand only a few alternations of quantifiers over a computable relation [23]. Indeed, we typically use at most $0'''$-injury techniques when we construct two or more different computable presentations of an algebraic structure. As a consequence, unless there is a pattern that we could iterate, the isomorphisms that we can handle are usually at most $0'''$. An elementary example of this phenomenon is the classical Mal'cev's construction of a $\mathbb{Q}$-vector space in which linear independence is undecidable [20]. The standard "nice" and the Mal'cev's "complicated" presentations are isomorphic relative to the halting problem $0'$. In fact, *any* two computable copies of this vector space are isomorphic relative to $0'$. A non-elementary example is a remarkable result of Goncharov, Molokov and Romanovskiĭ [13] (based on Goncharov [12]) saying that there exists a computable, infinitely generated nilpotent group with *exactly two* computable presentations up to computable isomorphism. This bizarre nilpotent group has a unique computable representation up to $0''$-isomorphism. It is not known whether this upper bound on the complexity of isomorphism could be improved to some $\mathbf{a} <_T 0''$. It is known, however, that if there exist two computable presentations of a structure that are $0'$-isomorphic but not computably isomorphic, then the structure has infinitely many computable presentations up to computable isomorphism [11,9]. For instance, many abelian groups have this property [11]. We refer to [1,9] for more examples of this nature.

Seeking a deeper understanding of these and similar constructions, we would like to accumulate more knowledge about computable structures isomorphic relative to a few iterations of the halting problem. The definition below was suggested by Ash.

Definition 1.1. A computable algebraic structure A is Δ^0_n-categorical if every two computable presentations of A are $\emptyset^{(n-1)}$-isomorphic.

Clearly, Δ^0_n-categoricity is a natural generalization of computable categoricity (set $n = 1$), and thus the notion is interesting on its own right. Ash [2] was the first to systematically study Δ^0_n-categorical computable structures. He described Δ^0_n-categorical well-orders. Although there are several further deep results on Δ^0_n-categorical structures in the literature ([3,21,8], see also Chapter 17 of [1]), our understanding of Δ^0_n-categoricity is rather limited even when $n = 2$. While computable categoricity was characterized for Boolean algebras, linear orders, torsion-free abelian groups and many other standard classes [1,9], we don't have a satisfactory description of Δ^0_2-categoricity in any of these classes. As it seems, Δ^0_2-categoricity is far less well-behaved than computable categoricity. For instance, in contrast to computable categoricity, Δ^0_2-categoricity tends to be different from *relative* Δ^0_2*-categoricity*[2] already in rather simple algebraic classes [4,16,5]. As a consequence, the study of Δ^0_2-categoricity usually requires new algebraic and computability-theoretic ideas (e.g., [7]), and thus such investigations are of some technical interest as well.

1.1. Complex isomorphisms between simple structures

Our intention is to study Δ^0_2-categoricity and Δ^0_2-isomorphisms within an algebraic context which is as simple as possible. We would like to pick a class where algebra would not be the main obstacle (in contrast to, say, [21,7]) and concentrate on the *computability-theoretic* combinatorics of Δ^0_2-isomorphisms.

[1] As usual, $0^{(n+1)}$ stands for the n'th iteration of the halting problem, up to Turing equivalence. More generally, the *Turing jump operator* $X \to X'$ is well-defined up to Turing equivalence on arbitrary oracles $X \subseteq \mathbb{N}$.

[2] Recall that a computable structure $\mathcal{B}$ is relatively Δ^0_n-categorical if the $(n-1)'$th Turing jump $D_0(\mathcal{A})^{(n-1)}$ of the open diagram $D_0(\mathcal{A})$ of $\mathcal{A} \cong \mathcal{B}$ computes an isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Note $\mathcal{A}$ does not have to be computable.