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Multi-posets in algebraic logic, group theory, and non-commutative topology ☆

Wolfgang Rump

Institute for Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, D-70550 Stuttgart, Germany

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ABSTRACT

The recent discovery that very different types of algebras have a quantale as an injective envelope is analyzed. Multi-posets are introduced as a generic structure that admits an essential embedding into a quantale, explicitly realized as a completion. Effect algebras and their non-commutative extensions, quantum B-algebras, KL-algebras, groups, and various other types of algebras are genuine multi-posets, in the sense that they determine full subcategories. Some structures like partially ordered groups or effect algebras do not carry over to the injective envelope. We provide a criterion for extendability in terms of a generalized archimedean property. Applied to non-commutative topology, multi-posets lead to a symmetric version of quantum spaces as a class of spatial quantales.

0. Introduction

The point of depart for this investigation was a paper of Lambek et al. [24] where the injective envelope of a partially ordered monoid is obtained as a quantale. The construction is remarkable in that it depends on residuals, that is, left and right adjoint operations of the multiplication, in an essential way. These adjoint operations $\to$ and $\rightsquigarrow$ can be interpreted logically as left and right implications, falling apart in a setting where commutativity of the conjunction is waived. Now if a partially ordered semigroup M is embedded into

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E-mail address: rump@mathematik.uni-stuttgart.de.

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an arbitrary quantale, the closure with respect to the residuals is a stable structure, a *quantum B-algebra* [39,41], which does not depend on the chosen embedding, and the injective envelope of M is just the *completion*, or equivalently, the injective envelope, of the quantum B-algebra X_M generated by M [40].

Due to the adjunction between multiplication and its residuals, quantum B-algebras are just the implicational counterpart of partially ordered semigroups. They can be characterized as subposets and implicational subreducts of quantales ([39], Theorem 2.3). In other words, a system $(X; \cdot, \leqslant)$ embeds into a quantale if and only if it is a partially ordered semigroup, while $(X; \to, \rightsquigarrow, \leqslant)$ embeds into a quantale if and only if it is a *quantum B-algebra*, that is,

$$x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z) \tag{1}$$

$$x \leqslant y \to z \iff y \leqslant x \rightsquigarrow z \tag{2}$$

$$y \leqslant z \implies x \to y \leqslant x \to z \tag{3}$$

hold for $x, y, z \in X$. For various examples, see [39,41]. Here we only mention that although there is no explicit multiplication or unit element, a quantum B-algebra with trivial partial order is equivalent to a group (see [39], Theorem 4.2). This points to hidden operations and structures which are not apparent from the few statements (1)–(3).

In the present paper, we push this aspect further by showing that the result of Lambek et al. [24] and its counterpart for quantum B-algebras [40] are special cases of an embedding theorem for a more fundamental structure with no operations at all. Such a structure M is given by a single relation

$$x_1 \cdots x_n \leqslant x \tag{4}$$

among $n+1$ elements $x, x_1, \dots, x_n \in M$ with varying $n > 0$, so that any further relation $y_1 \cdots y_m \leqslant x_i$ implies that (4) remains valid if x_i is replaced by $y_1 \cdots y_m$. If, in addition, $x \leqslant y \leqslant x$ is equivalent to $x = y$, we call M a *multi-poset*. Thus, from a logical point of view, multi-posets formalize the basic property of the entailment relation $\leqslant$ in several variables, namely, (4) states that the conjunction of $x_1, \dots, x_n$ (in this order!) entails x, without assuming that the left-hand side of (4) represents an element of M.

On the other hand, quite surprisingly, partially ordered semigroups as well as quantum B-algebras are genuine multi-posets, which means that in both cases, a relation (4) can be defined such that the category of partially ordered semigroups (with the morphisms given in [24]), and the category of quantum B-algebras, are full subcategories of the category **mPos** of multi-posets (with the obvious morphisms). In particular, this implies – *pars pro toto* – that a full subcategory of **mPos** is formed by the category **Grp** of groups.

The concept of embedding, which looks a bit unwieldy for partially ordered semigroups as well as for quantum B-algebras, becomes quite natural in the extended framework of multi-posets. In this respect, multi-posets appear to be inevitable, and it astounds that they have not been studied before. Every multi-poset has a completion which coincides with its injective envelope (Theorem 2). Injective multi-posets are the same as quantales. Calculations with multi-posets can thus be conveniently performed within a quantale.

Lattice-ordered groups G can be regarded as multi-posets in at least three ways. As a quantum B-algebra, G itself is a multi-poset. Secondly, the negative cone G^- is a quantum B-algebra, with $x \to y := yx^{-1} \wedge 1$. Furthermore, G^- is a *KL-algebra* [37], satisfying the equation

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$$

which is fundamental in the theory of braces and the quantum Yang–Baxter equation [35,36]. KL-algebras are multi-posets (Proposition 2), but not always quantum B-algebras. We provide simple criteria to charac-