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A separating hyperplane theorem, the fundamental theorem of asset pricing, and Markov's principle

Josef Berger *, Gregor Svindland

Department of Mathematics, LMU Munich, Theresienstraße 39, D-80333 München, Germany

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ABSTRACT

We prove constructively that every uniformly continuous convex function $f : X \to \mathbb{R}^+$ has positive infimum, where X is the convex hull of finitely many vectors. Using this result, we prove that a *separating hyperplane theorem*, the *fundamental theorem of asset pricing*, and *Markov's principle* are constructively equivalent. This is the first time that important theorems are classified into Markov's principle within constructive reverse mathematics.

Constructive mathematics in the tradition of Errett Bishop [2,3] is characterised by not using the law of excluded middle as a proof tool. As a major consequence, properties of the real number line like the *limited principle of omniscience*

LPO $\quad \forall x, y \in \mathbb{R}\,(x < y \vee x > y \vee x = y)$,

the *lesser limited principle of omniscience*

LLPO $\quad \forall x, y \in \mathbb{R}\,(x \leq y \vee x \geq y)$,

* Corresponding author.
E-mail address: jberger@math.lmu.de (J. Berger).

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and *Markov's principle*

MP $\quad \forall x \in \mathbb{R}\,(\neg\,(x = 0) \Rightarrow |x| > 0)$

are no longer provable propositions but rather considered additional axioms.

In this context, *reverse mathematics* attempts on deciding which axioms of this kind are necessary and sufficient to prove certain theorems. For example, the Hahn–Banach theorem is equivalent to LLPO [8]. Many properties of the reals still hold constructively, and will be freely used in the sequel:

- $x \geq y \Leftrightarrow \neg\,(x < y)$
- $x = y \Leftrightarrow \neg\neg\,(x = y)$
- $|x| \cdot |y| > 0 \Leftrightarrow |x| > 0 \;\&\; |y| > 0$ (1)
- $|x| > 0 \Leftrightarrow x > 0 \vee x < 0$

We cannot prove constructively that every nonempty bounded set of reals has an infimum. This gives rise to the following definition. Fix $\varepsilon > 0$ and sets $D \subseteq C \subseteq X$ where (X, d) is a metric space. The set D is an *ε-approximation* of C if for every $x \in C$ there exists $y \in D$ with $d(x, y) < \varepsilon$. C is *totally bounded* if for every n there exist elements $x_1, \ldots, x_m$ of C such that $\{x_1, \ldots, x_m\}$ is a $1/n$-approximation of C. In particular, any inhabited[1] totally bounded subset C of X is *located* [5, Proposition 2.2.9], which means that

$$d(x, C) = \inf\{d(x, y) \mid y \in C\}$$

exists for all $x \in X$. If C is totally bounded, and $f : C \to \mathbb{R}$ is uniformly continuous, then

$$\inf f = \inf\{f(y) \mid y \in C\}$$

does exist [5, Corollary 2.2.7]. In this context, Brouwer's *fan theorem* can be stated as follows [10].

FAN $\quad$ If $f : [0, 1] \to \mathbb{R}^+$ is uniformly continuous, then $\inf f > 0$.

Note that FAN can be deduced from LLPO [9]. There are many propositions which are equivalent to principles like LLPO or the fan theorem. In this paper, for the first time, we prove that important classical theorems are classified into Markov's principle.

Set

$$\mathcal{Y}_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i > 0 \text{ and } 0 \leq x_i \text{ for all } i\right\},$$

$$\mathcal{X}_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1 \text{ and } 0 \leq x_i \text{ for all } i\right\},$$

and

$$\mathcal{P}_n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 1 \text{ and } 0 < x_i \text{ for all } i\right\}.$$

[1] A set is *inhabited* if it contains an element, which is classically equivalent to being nonempty.