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The tree property below $\aleph_{\omega\cdot 2}$

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ABSTRACT

We improve the best known result on successive regular cardinals with the tree property. In particular we prove that relative to an increasing $\omega + \omega$-sequence of supercompact cardinals it is consistent that every regular cardinal on the interval $[\aleph_2, \aleph_{\omega\cdot 2})$ has the tree property.

In this paper we improve the best known partial result towards answering an old question of Magidor, "Is it consistent that every regular cardinal greater than $\aleph_1$ has the tree property?". Neeman [6] has shown that starting from ω supercompact cardinals one can force to obtain the tree property at every regular cardinal in the interval $[\aleph_2, \aleph_{\omega+1}]$. In this paper we prove the following theorem.

Theorem 0.1. *Assuming there is an increasing $\omega + \omega$-sequence of supercompact cardinals, then there is a generic extension in which every regular cardinal in the interval $[\aleph_2, \aleph_{\omega\cdot 2})$ has the tree property.*

The history of these results goes back to a theorem of Mitchell and Silver [5], who showed it is consistent relative to a weakly compact cardinal that $\aleph_2$ has the tree property.

Abraham [1] showed relative to a supercompact cardinal with a weakly compact cardinal above it is consistent that $\aleph_2$ and $\aleph_3$ have the tree property. Cummings and Foreman [2] improved this to show that it is consistent that all $\aleph_n$'s for $n \geq 2$ have the tree property from ω supercompact cardinals. Obtaining the tree property at $\aleph_{\omega+1}$ was first done by Magidor and Shelah [4] from very large cardinals. Recently Sinapova [7] was able to reduce the upper bound on the strength of the tree property at $\aleph_{\omega+1}$ to just ω supercompact cardinals using a Prikry type construction. Broadly speaking, Neeman's result mentioned

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above can be seen as combining ideas from the results of Cummings and Foreman, and Sinapova. In this paper we push the ideas from Neeman's result further.

An important remark is that in the model for the main theorem $\aleph_\omega$ is not strong limit, in particular $2^{\omega_1} = \aleph_{\omega+2}$. In doing so we avoid a difficult question of Woodin's from the 1980s which asks whether it is consistent to have the failure of both SCH at $\aleph_\omega$ and the tree property at $\aleph_{\omega+1}$. See [8] for a recent partial result. Woodin's question does not mention anything about the combinatorics below $\aleph_\omega$ and current techniques seem far from being able to combine a forcing like the one of Cummings and Foreman with a Prikry type construction in a satisfactory way. We refer the reader to [11] for such an attempt.

The result makes use of the poset and many of the main lemmas from Neeman's paper [6]. The reader is advised to have a copy of it on hand. Throughout the paper we have attempted to keep the notation very close to Neeman's. One notable convention is the use of τ-closed to mean every decreasing sequence of length τ has a lower bound.

The paper is organized into sections based on these topics: preliminaries, definition of the main forcing, cardinal structure, the tree property below $\aleph_\omega$, the tree property at $\aleph_{\omega+1}$, the tree property at $\aleph_{\omega+2}$, and the tree property above $\aleph_{\omega+2}$.

1. Preliminaries

We list some essential lemmas which will be used in the proof below. The first two are preservation lemmas due to the author.

Lemma 1.1. *(See [10].) If $\mathbb{P} \times \mathbb{P}$ is κ-cc, then forcing with $\mathbb{P}$ cannot add a branch through a tree of height κ.*

Lemma 1.2. *(See [9].) If $V \subseteq V'$ is a τ^+-cc forcing extension such that $2^\tau \geq \eta$ in V and $\mathbb{P}$ is τ-closed in V, then forcing with $\mathbb{P}$ over V' cannot add a branch through an η-tree.*

We will also make use of a lemma of Abraham [1], which allows us to preserve the chain condition of certain posets. We take our statement of the lemma from Cummings and Foreman [2].

Lemma 1.3. *Let $\tau < \kappa$ and assume $V \vDash$ "τ is regular and κ is inaccessible." Let $\mathbb{P} = \mathrm{Add}(\tau, \eta)_V$. If $W \supseteq V$ is a model of set theory where*

(1) *τ and κ are cardinals in W and*
(2) *every set of ordinals in W of size less than κ is covered by a set of ordinals in V of size less than κ,*

then $\mathbb{P}$ has the κ-Knaster property in W, and in particular its square is κ-cc.

Remark 1.4. The above lemma is often stronger than what we need. We will often need to see that the square of a Cohen poset retains some chain condition property in an outer model. However a Cohen poset is isomorphic to its square. So for an application of Lemma 1.1, it is enough to show that the chain condition of the Cohen poset is preserved in an outer model.

We also need Easton's Lemma [3].

Lemma 1.5. *If $\mathbb{P}$ is τ-cc and $\mathbb{R}$ is $< \tau$-closed, then it is forced by $\mathbb{P}$ that $\mathbb{R}$ is $< \tau$-distributive.*