Annals of Pure and Applied Logic 167 (2016) 262–283

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Counterexamples to countable-section Π^1_2 uniformization and Π^1_3 separation ☆

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ARTICLE INFO

Article history:
Received 27 March 2015
Accepted 11 November 2015
Available online 28 December 2015

MSC:
03E15
03E35
03E20
28A05

Keywords:
Uniformization
Separation
Jensen's forcing
Finite support

ABSTRACT

We make use of a finite support product of the Jensen minimal Π^1_2 singleton forcing to define a model in which Π^1_2 uniformization fails for a set with countable cross-sections. We also define appropriate submodels of the same model in which separation fails for Π^1_3.

1. Introduction

The uniformization problem, introduced by Luzin [17,18], is well known in modern set theory. (See Moschovakis [19], Kechris [16], Hauser and Schindler [6] for both older and more recent studies.) In particular, it is known that every $\mathbf{\Sigma}^1_2$ set can be uniformized by a set of the same class $\mathbf{\Sigma}^1_2$, but on the other hand, there is a $\mathbf{\Pi}^1_2$ set (in fact, a lightface Π^1_2 set), not uniformizable by any set in $\mathbf{\Pi}^1_2$. The negative part of this result cannot be strengthened much further in the direction of the absence of more complicated uniformizing sets since any $\mathbf{\Pi}^1_2$ set admits a $\mathbf{\Delta}^1_3$-uniformization assuming $\mathbf{V} = \mathbf{L}$ and admits a $\mathbf{\Pi}^1_3$-uniformization assuming the existence of sharps (the Martin–Solovay–Mansfield theorem, [19, 8H.10]).

☆ This document is a collaborative effort.

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1 Partial support of RFFI grant 13-01-00006 acknowledged.
2 Partial support of the Russian Science Foundation grant 14-50-00150 acknowledged.

http://dx.doi.org/10.1016/j.apal.2015.12.002

However, the mentioned $\mathbf{\Pi}^1_2$-non-uniformization theorem can be strengthened in the context of consistency. For instance, the Π^1_2 set

$$P = \{\langle x, y\rangle \,:\, x, y \in 2^\omega \wedge y \notin \mathbf{L}[x]\}$$

is not uniformizable by any ROD (real-ordinal definable) set in the Solovay model and many other models of **ZFC** in which it is not true that $\mathbf{V} = \mathbf{L}[x]$ for a real x, and then the cross-sections of P can be considered as "large", in particular, they are definitely uncountable. Therefore one may ask:

Question 1. Is it consistent that there is a ROD-non-uniformizable Π^1_2 set P such that all cross-sections $P_x = \{y \,:\, \langle x, y\rangle \in P\}$ are at most countable?

This question is obviously connected with another question, initiated and briefly discussed at the Mathoverflow exchange desk[3] and at FOM[4]:

Question 2. Is it consistent with **ZFC** that there is a *countable* definable set of reals $X \neq \varnothing$ which has no OD (ordinal definable) elements.

Ali Enayat (footnote 4) conjectured that Question 2 can be solved in the positive by the finite-support product $\mathbb{P}^{<\omega}$ of countably many copies of the Jensen "minimal Π^1_2 real singleton forcing" $\mathbb{P}$ defined in [9].[5] Enayat demonstrated in [2] that a symmetric part of the $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$, the constructible universe, definitely yields a model of **ZF** (not a model of **ZFC**!) in which there is a Dedekind-finite infinite OD set of reals with no OD elements.

Following the mentioned conjecture, we proved in [14] that indeed it is true in a $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$ that the set of $\mathbb{P}$-generic reals is a countable non-empty Π^1_2 set with no OD elements.[6] Using a finite-support product $\prod_{\xi<\omega_1} \mathbb{P}_\xi{}^{<\omega}$, where the forcing notions $\mathbb{P}_\xi$ are pairwise different clones of Jensen's forcing $\mathbb{P}$, we answer Question 1 in the positive.

Theorem 1.1. *In a suitable generic extension of $\mathbf{L}$, it is true that there is a lightface Π^1_2 set $P \subseteq 2^\omega \times 2^\omega$ whose all cross-sections $P_x = \{y \,:\, \langle x, y\rangle \in P\}$ are at most countable, but P is not uniformizable by a ROD set.*

Using an appropriate generic extension of a submodel of the same model, similar, to some extent, to models considered in Harrington's unpublished notes [5], we also prove

Theorem 1.2. *In a suitable generic extension of $\mathbf{L}$, it is true that there is a pair of disjoint lightface Π^1_3 sets $X, Y \subseteq 2^\omega$, not separable by disjoint $\mathbf{\Sigma}^1_3$ sets, and hence $\mathbf{\Pi}^1_3$ Separation and Π^1_3 Separation fail.*

This result was first proved by Harrington in [5] on the basis of almost disjoint forcing of Jensen–Solovay [10], and in this form has never been published, but was mentioned in [19, 5B.3] and [7, page 230]. A complicated alternative proof of Theorem 1.2 can be obtained with the help of *countable-support* products and iterations of Jensen's forcing studied earlier in [1,11,12]. The *finite-support* approach which we pursue here

[3] A question about ordinal definable real numbers. Mathoverflow, March 09, 2010. http://mathoverflow.net/questions/17608.

[4] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html.

[5] *Jensen's forcing* below, for the sake of brevity—on this forcing, see also 28A in [8].

[6] We also proved in [15] that the existence of a Π^1_2 E_0-class with no OD elements is consistent with **ZFC**, using a E_0-invariant version of the Jensen forcing. A related consistency result on countable Groszek–Laver pairs, established by similar methods, will appear in [3].