Annals of Pure and Applied Logic 167 (2016) 309–334

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

# Iterated elementary embeddings and the model theory of infinitary logic

John T. Baldwin [a,1], Paul B. Larson [b,*,2]

[a] *Department of Mathematics, Statistics, and Computer Science M/C 249, University of Illinois at Chicago, 851 S. Morgan, Chicago, IL 60607, United States*
[b] *Department of Mathematics, Miami University, Oxford, OH 45056, United States*

ARTICLE INFO

*Article history:*
Received 21 March 2015
Accepted 11 December 2015
Available online 28 December 2015

*MSC:*
03C48
03E15
03E57
03C45

*Keywords:*
Abstract elementary classes
Galois types
Iterated elementary embeddings
Absoluteness

ABSTRACT

We use iterations of elementary embeddings derived from countably complete ideals on $\omega_1$ to provide a uniform proof of some classical results connecting the number of models of cardinality $\aleph_1$ in various infinitary logics to the number of syntactic types over the empty set. We introduce the notion of an analytically presented abstract elementary class (AEC), which allows the formulation and proof of generalizations of these results to refer to Galois types rather than syntactic types. We prove (Theorem 0.4) the equivalence of $\aleph_0$-presented classes and analytically presented classes and, using this, generalize (Theorem 0.5) Keisler's theorem on few models in $\aleph_1$ to bound the number of Galois types rather than the number of syntactic types. Theorem 0.6 gives a new proof (cf. [5]) for analytically presented AEC's of the absoluteness of $\aleph_1$-categoricity from amalgamation in $\aleph_0$ and almost Galois $\omega$-stability.



This paper combines methods of axiomatic and descriptive set theory to study problems in model theory. In particular, we use iterated generic elementary embeddings to analyze the number of models in $\aleph_1$ in various infinitary logics and for Abstract Elementary Classes (AEC). The technique here provides a uniform method for approaching and extending theorems that Keisler et al. proved in the 1970's relating the existence of uncountable models realizing many types to the existence of many models in $\aleph_1$ (Theorem 0.3). To formalize this uniformity we introduce the notion of an analytically presented AEC and show that it is a further but more useful variant for the well-known notion of $PC_\delta$ over $L_{\omega_1,\omega}$ (Theorem 3.3). This allows us to extend the Keisler-style results relating the number of types in $\aleph_0$ to the number of models in $\aleph_1$ from syntactic types to Galois types (Theorem 5.8). Finally, we prove some absoluteness results leading to the

* Corresponding author.
*E-mail addresses:* jbaldwin@uic.edu (J.T. Baldwin), larsonpb@miamioh.edu (P.B. Larson).
[1] Partially supported by a Simons Foundation travel grant G5402.
[2] Supported in part by NSF grants DMS-0801009 and DMS-1201494.

http://dx.doi.org/10.1016/j.apal.2015.12.004
0168-0072/

conclusion that categoricity in $\aleph_1$ is absolute for analytically presented AEC that satisfy the amalgamation property in $\aleph_0$ and are almost Galois-$\omega$-stable (Theorem 0.6).

The arguments presented here are related to arguments by Farah, Larson, et al. in [11,12], in which iterated elementary embeddings were used to prove forcing-absoluteness results. Those papers focused on the large cardinal context. Here we work primarily in ZFC. Model-theoretic inductions to iterated generic elementary embeddings can be found in [25] and [6].

The approach of this paper is the following proof scheme which can be abbreviated as *consistency yields provability*: Prove that a model theoretic property $\Phi$ holds in a model $N$ of ZFC$^\circ$. Of course if $\Phi$ is absolute between $N$ and $V$ this is an ancient technique to attain provability. But, here we extend the model $N$ by ultralimits to (one or many) models $N^*$ satisfying $\Phi$ and such that $\Phi$ is absolute between $N^*$ and $V$.

We refer the reader to [36,1] for model-theoretic definitions such as amalgamation and Shelah's notion of *Abstract Elementary Class*, and for background on the notions used here. We use [1] as single reference with a uniform notation containing many results of Keisler, Shelah and others. Similarly, Gao's text [15] is used for descriptive set theory. For example, Theorem 0.2 is stated for atomic models of first order theories. The equivalence between the atomic model context and models of a complete sentence in $L_{\omega_1,\omega}$ is explained in Chapter 6 of [1]. Abstract Elementary Classes form a general context unifying many of the properties of such infinitary logics as $L_{\omega_1,\omega}$, $L_{\omega_1,\omega}(Q)$.

A fundamental result in the study of $\aleph_1$-categoricity for Abstract Elementary Classes is the following theorem of Shelah (see [1], Theorem 17.11).

**Fact 0.1** *(Shelah).* Suppose that $\mathbf{K}$ is an Abstract Elementary Class such that

- The Lówenheim–Skolem number, $\mathrm{LS}(\mathbf{K})$, is $\aleph_0$;
- $\mathbf{K}$ is $\aleph_0$-categorical;
- amalgamation fails[3] for countable models in $\mathbf{K}$.

Suppose also that $2^{\aleph_0} < 2^{\aleph_1}$. Then there are $2^{\aleph_1}$ non-isomorphic models of cardinality $\aleph_1$ in $\mathbf{K}$.

Theorem 0.1 is one of the two fundamental tools to develop the stability theory of $L_{\omega_1,\omega}$. The second is the following theorem of Keisler (see [25,1], Theorem 5.2.5).

**Fact 0.2** *(Keisler).* If a $PC_\delta$ over $L_{\omega_1,\omega}$ class $\mathbf{K}$ has an uncountable model but less than $2^{\omega_1}$ models of power $\aleph_1$, then for any countable fragment $L_{\mathcal{A}}$, *every* member of $\mathbf{K}$ realizes only countably many $L_{\mathcal{A}}$-types over $\emptyset$.

The notion of $\omega$-stability for sentences in $L_{\omega_1,\omega}$ is a bit subtle and is more easily formulated for the associated class $\mathbf{K}$ of atomic models of a first theory with first order elementary embedding as $\prec_{\mathbf{K}}$. For *countable* $A \subseteq M \in \mathbf{K}$, $S_{at}(A)$ denotes the set of first order types over $A$ realized in atomic models.[4] $\mathbf{K}$ is $\omega$-stable[5] if for each countable $M \in \mathbf{K}$, $|S_{at}(M)| = \aleph_0$.

Combining these two theorems, Shelah showed (under the assumption $2^{\aleph_0} < 2^{\aleph_1}$) that a complete sentence of $L_{\omega_1,\omega}$ which has less than $2^{\aleph_1}$ models in $\aleph_1$ has the amalgamation property in $\aleph_0$ and is $\omega$-stable. Crucially, Shelah's argument relies on the assumption $2^{\aleph_0} < 2^{\aleph_1}$ in two ways. It first uses a variation of the Devlin–Shelah weak diamond principle [9] for Theorem 0.1. Then using amalgamation, extending Keisler's theorem from types over the empty set to types over a countable model is a straightforward counting

[3] Unlike first order logic, this is a strictly stronger statement than 'amalgamation fails over subsets of models of $\mathbf{K}$.'

[4] This definition does not extend to uncountable $A$, see page 138 of [1].

[5] This requirement that $M$ is a model is essential; Example 3.17 of [1], covers of the multiplicative group of $\mathbb{C}$, is $\omega$-stable but there are countable atomic $A$ with $|S_{at}(A)| = 2^{\aleph_0}$.