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Reducts of the generic digraph

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ABSTRACT

The generic digraph (D, E) is the unique countable homogeneous digraph that embeds all finite digraphs. In this paper, we determine the lattice of reducts of (D, E), where a structure $\mathcal{M}$ is a reduct of (D, E) if it has domain D and all its $\emptyset$-definable relations are $\emptyset$-definable relations of (D, E). As (D, E) is $\aleph_0$-categorical, this is equivalent to determining the lattice of closed groups that lie in between $\mathrm{Aut}(D, E)$ and $\mathrm{Sym}(D)$.

This paper is a part of a large body of work concerning reducts of first-order structures, where $\mathcal{N}$ is said to be a reduct of $\mathcal{M}$ if all $\emptyset$-definable relations in $\mathcal{N}$ are $\emptyset$-definable in $\mathcal{M}$. A common set-up is that one studies the reducts of some given structure $\mathcal{M}$, where two reducts which are interdefinable are considered to be equal. These reducts form a lattice and when the structure is $\aleph_0$-categorical this is equivalent to studying the lattice of closed subgroups lying between $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Sym}(M)$.

The first results in this area were the classification of the reducts of $(\mathbb{Q}, <)$ [5] and of the random graph Γ [12]. In [13], Thomas conjectured that all homogeneous structures in a finite relational language have only finitely many reducts. This question remains unsolved and continues to provide motivation for study. More recent results include the classification of the reducts of $(\mathbb{Q}, <, 0)$ [8] and of the affine and projective spaces over $\mathbb{Q}$ [9].

A surprising development in this area is the connection with constraint satisfaction in complexity theory, by Bodirsky and Pinsker. This connection is made via clone theory in universal algebra. In order to analyse certain closed clones they developed a Ramsey-theoretic tool, named 'canonical functions'. With further

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developments [2,4], canonical functions now provide a powerful tool in studying reducts, for example, they were used to classify the reducts of the generic partial order [11] and of the generic ordered graph [3].

In this paper, we determine the lattice of reducts of the generic directed graph, which we denote by (D, E). For us, a directed graph (or digraph) means a set of vertices with directed edges between them, where we do *not* allow an edge going in both directions. The generic digraph is the unique countable homogeneous digraph that embeds all finite digraphs. 'Homogeneous' means that every isomorphism $f : A \to B$, where $A, B \subset D$ are finite, can be extended to an automorphism of (D, E).

We outline the structure of the paper. In Section 1, we provide the necessary preliminary definitions and facts about the generic digraph and about reducts. We also comment on some notational conventions that we use. In Section 2, we define the reducts of the generic graph and provide the lattice, $\mathcal{L}$, that these reducts form. The main theorem is that this lattice $\mathcal{L}$ is the lattice of all the reducts of the generic digraph. In Section 3, we describe the reducts in some detail, establishing notation and important lemmas that are used in the rest of the paper. In Section 4, we show that $\mathcal{L}$ is indeed a sublattice of the lattice of reducts. In Section 5, we prove that $\mathcal{L}$ does contain all the reducts of (D, E). The section starts by describing the information that is obtained from the known classifications of the random graph and the random tournament [1]. We then give the background definitions and results on canonical functions at the start of Section 5.2, and we also carry out the combinatorial analysis of the canonical functions in this section. Section 5 ends by using the analysis to complete the proof of the main theorem. In Section 6, we provide a summary and some open questions.

1. Preliminaries

1.1. Notational conventions

We sometimes write 'ab' as an abbreviation for (a, b), e.g., we may write "Let ab be an edge of the digraph D". Structures are denoted by $\mathcal{M}, \mathcal{N}$, and their domains are M and N respectively. If $A \subseteq M$, A^c denotes the complement of A. $\mathrm{Sym}(M)$ is the set of all bijections $M \to M$ and $\mathrm{Aut}(\mathcal{M})$ is the set of all automorphisms of $\mathcal{M}$. Given a formula $\phi(x, y)$, we use $\phi^*(x, y)$ to denote the formula $\phi(y, x)$. $S(\mathcal{M})$ denotes the space of types of the theory of $\mathcal{M}$. If f has domain A and $(a_1, \ldots, a_n) \in A^n$, then $f(a_1, \ldots, a_n) := (f(a_1), \ldots, f(a_n))$. For $\bar{a}, \bar{b} \in M^n$, we say $\bar{a}$ and $\bar{b}$ are isomorphic, and write $\bar{a} \cong \bar{b}$, to mean that the function $a_i \mapsto b_i$ for all i such that $1 \le i \le n$ is an isomorphism.

There will be instances where we do not adhere to strictly correct notational usage, however, the meaning will be clear from the context. For example, we may write '$a \in (a_1, \ldots, a_n)$' instead of '$a = a_i$ for some i such that $1 \le i \le n$'. Another example is that we sometimes use c to represent the singleton set $\{c\}$ containing it. A third example is we may write '$\bar{a} \in A$' instead of '$\bar{a} \in A^n$ for some n'.

1.2. The generic digraph

Definition 1.1.

(i) A directed graph (V, E) consists of a set V and an irreflexive, antisymmetric relation $E \subseteq V^2$. V represents the set of vertices and E represents the set of directed edges, so if $(a, b) \in E$, we visualise it as an edge going out of a and into b. We abbreviate 'directed graph' by 'digraph'.

(ii) By an empty digraph we mean a digraph whose edge set is empty.

(iii) We say that a structure $\mathcal{M}$ is homogeneous if every isomorphism $f : A \to B$, where A, B are finite substructures of $\mathcal{M}$, can be extended to an automorphism of $\mathcal{M}$.

(iv) The generic digraph, which we denote by (D, E), is the unique (up to isomorphism) countable homogeneous digraph that embeds all finite digraphs.

(v) $N(x, y)$ will denote the non-edge relation of (D, E), so $N(x, y) := \neg E(x, y) \wedge \neg E^*(x, y)$.