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Propositional logics of dependence ☆

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ABSTRACT

In this paper, we study logics of dependence on the propositional level. We prove that several interesting propositional logics of dependence, including propositional dependence logic, propositional intuitionistic dependence logic as well as propositional inquisitive logic, are expressively complete and have disjunctive or conjunctive normal forms. We provide deduction systems and prove the completeness theorems for these logics.

1. Introduction

The idea of dependence logic, introduced in [30] on the basis of Hodges [18], is the following: If truth in the traditional sense, i.e. as Tarski defined it, is defined with respect to a *set* of assignments, rather than just *one* assignment, it becomes possible to talk meaningfully about variables being *dependent* or *independent* from each other. The set of assignments can be thought of as a *data set*, giving evidence—much as in statistics—about mutual dependencies between the variables. If the set of assignments—the data—is thought of as a *database* we arrive at the concept of *functional dependency*, an important concept in database theory since

☆ Most of the results in this paper were included in the dissertation of the first author [34], which was supervised by the second author.

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Codd's pioneering paper [8]. We can also consider the set of assignments as expressing *uncertainty* about one "true" assignment, as in *inquisitive logic* [7], or as indicating *belief* about an unknown assignment, as in *doxastic logic* [13]. Finally, considering truth as given by sets of assignments leads naturally to the concept of the *probability* that a randomly chosen assignment (from the given set) satisfies a given propositional formula. This idea is developed in [19] to analyze so-called Bell's Inequalities of Quantum Physics.

Following [30], we call sets of assignments *teams*. Teams have been previously used to study dependence concepts in predicate logic [30] and modal logic [31]. We now focus this study to propositional logic. The fundamental concept of dependence logic is the concept $=\!\!(\vec{x}, y)$ of a variable y depending on a sequence $\vec{x}$ of other variables, which is taken as a new atomic formula. The meaning of such atomic dependence formulas is given via *teams*.

Studying the logics of dependence concepts in propositional logic resembles the case of predicate logic in that we use the method of teams. A *team* in this case is defined to be *a set of valuations* of propositional variables. There are, however, also significant differences between the predicate logic and the propositional cases. Notably, propositional logics of dependence are *decidable*. This is because for any given formula of the logics with n propositional variables, there are in total 2^n valuations and 2^{2^n} teams. The method of truth tables has its analogue in these logics, but the size of such tables grows exponentially faster than in the case of classical propositional logic, rendering it virtually inapplicable. This emphasizes the role of the axioms and the completeness theorem in providing a manageable alternative for establishing logical consequence.

Classical propositional logic is based on propositions of the form

$$\begin{array}{l} p \\ \text{Not } p \\ p \text{ or } q \\ \text{If } p, \text{ then } q \end{array}$$

and more generally

$$\text{If } p_{i_1}, \dots, p_{i_k}, \text{ then } q. \tag{1}$$

We present extensions of classical propositional logic in which one can express, in addition to the above, propositions of the form "q depends on p", or more generally

$$q \text{ depends on } p_{i_1}, \dots, p_{i_k}. \tag{2}$$

In our setting, (2) is treated as an atomic fact. This is expressed formally by a new atomic formula

$$=\!\!(p_{i_1}, \dots, p_{i_k}, q), \tag{3}$$

which we call the *dependence atom*.

Intuitively, (2) means that to know whether q holds it is sufficient to consult the truth values of $p_{i_1}, \dots, p_{i_k}$. Note that, as in the first-order dependence logic case, (2) says nothing about the *way* in which $p_{i_1}, \dots, p_{i_k}$ are logically related to q. It may be that $p_{i_1} \wedge \dots \wedge p_{i_k}$ logical implies q, or that $\neg p_{i_1} \wedge \dots \wedge \neg p_{i_k}$ logical implies $\neg q$, or anything in between. Technically speaking, this is to say:

$$\text{The truth value of } q \text{ is a function of the truth values of } p_{i_1}, \dots, p_{i_k}. \tag{4}$$

Some examples of natural language sentences involving dependence are the following:

1. *Whether it rains depends completely on whether it is winter or summer.*
2. *Whether you end up in the town depends entirely on whether you turn left here or right.*
3. *I will be absent depending on whether he shows up or not.*