Annals of Pure and Applied Logic 167 (2016) 408–430

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Regularity properties on the generalized reals

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ARTICLE INFO

Article history:
Received 6 July 2014
Accepted 18 July 2015
Available online 10 February 2016

MSC:
03E15
03E40
03E30
03E05

Keywords:
Generalized Baire spaces
Regularity properties
Descriptive set theory

ABSTRACT

We investigate regularity properties derived from tree-like forcing notions in the setting of "generalized descriptive set theory", i.e., descriptive set theory on κ^κ and 2^κ, for regular uncountable cardinals κ.

1. Introduction

Generalized Descriptive Set Theory is an area of research dealing with generalizations of classical descriptive set theory on the Baire space ω^ω and Cantor space 2^ω, to the *generalized Baire space* κ^κ and the *generalized Cantor space* 2^κ, where κ is an uncountable regular cardinal satisfying $\kappa^{<\kappa} = \kappa$. Some of the earlier papers dealing with descriptive set theory on $(\omega_1)^{\omega_1}$ were motivated by model-theoretic concerns, see e.g. [24] and [30, Chapter 9.6]. More recently, generalized descriptive set theory became a field of interest in itself, with various aspects being studied for their own sake, as well as for their applications to different fields of set theory.

This paper is the first systematic study of *regularity properties* for subsets of generalized Baire spaces. We will focus on regularity properties derived from tree-like forcing partial orders, using the framework introduced by Ikegami in [16] (see Definition 3.1) as a generalization of the Baire property, as well as a num-

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1 Supported by the Austrian Science Fund (FWF) under project numbers P23316 and P24654.
2 Supported by the Austrian Science Fund (FWF) under project number P23316.
3 Supported by the Austrian Science Fund (FWF) under project number P24654.

http://dx.doi.org/10.1016/j.apal.2016.01.001
0168-0072/

ber of other standard regularity properties (Lebesgue measurability, Ramsey property, Sacks property etc.) In the classical setting, such properties have been studied by many people, see, e.g., [15,3,4,19]. Typically, these properties are satisfied by analytic sets, while the Axiom of Choice can be used to provide counterexamples. On the second projective level one obtains independence results, as witnessed by "Solovay-style" characterization theorems, such as the following:

Theorem 1.1. *(See Solovay [29].) All $\mathbf{\Sigma}^1_2$ sets have the Baire property if and only if for every $r \in \omega^\omega$ there are co-meager many Cohen reals over $L[r]$.*

Theorem 1.2. *(See Judah–Shelah [15].) All $\mathbf{\Delta}^1_2$ sets have the Baire property if and only if for every $r \in \omega^\omega$ there is a Cohen real over $L[r]$.*

These types of theorems make it possible to study the relationships between different regularity properties on the second level. Far less is known for higher projective levels, although some results exist in the presence of large cardinals (see [16, Section 5]) and some other results can be found in [1, Chapter 9] and in the recent works [8,6]. Solovay's model [29] provides a uniform way of establishing regularity properties for all projective sets, starting from ZFC with an inaccessible.

When attempting to generalize descriptive set theory from ω^ω to κ^κ for a regular uncountable κ, at first many basic results remain intact after a straightforward replacement of ω by κ. But, before long, one starts to notice fundamental differences: for example, the generalized $\mathbf{\Delta}^1_1$ sets are not the same as the generalized Borel sets; absoluteness theorems, such as Σ^1_1- and Shoenfield absoluteness, are not valid; and in the constructible universe L, there is a $\mathbf{\Sigma}^1_1$-good well-order of κ^κ, as opposed to merely a $\mathbf{\Sigma}^1_2$-good well-order in the standard setting (see Section 2 for details). Not surprisingly, regularity properties also behave radically different in the generalized context. Halko and Shelah [13] first noticed that on 2^κ, the generalized Baire property provably fails for $\mathbf{\Sigma}^1_1$ sets. On the other hand, it holds for the generalized Borel sets, and is independent for generalized $\mathbf{\Delta}^1_1$ sets. This suggests that some of the classical theory on the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ level corresponds to the $\mathbf{\Delta}^1_1$ level in the generalized setting.

It should be noted that other kinds of regularity properties have been considered before, sometimes leading to different patterns in terms of consistency of projective regularity. For example, in [27] Schlicht shows that it is consistent relative to an inaccessible that a version of the perfect set property holds for all generalized projective sets. By [22], as well as recent results of Laguzzi and the first author, similar results hold for suitable modifications of the properties studied here.

This paper is structured as follows: Section 2 will be devoted to a brief survey of facts about the "generalized reals". In Section 3 we introduce an abstract notion of regularity and prove that, under certain assumption, the following results hold:

1. Borel sets are "regular".
2. Not all analytic sets are "regular".
3. For $\mathbf{\Delta}^1_1$ sets, the answer is independent of ZFC.

In Section 4 we focus on some concrete examples on the $\mathbf{\Delta}^1_1$-level and generalize some classical results from the $\mathbf{\Delta}^1_2$-level. Section 5 ends with a number of open questions.

2. Generalized Baire spaces

We devote this section to a survey of facts about κ^κ and 2^κ which will be needed in the rest of the paper, as well as specifying some definitions and conventions. None of the results here are new, though some are not widely known or have not been sufficiently documented.