Annals of Pure and Applied Logic 167 (2016) 615–648

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Modal logics, justification logics, and realization

Melvin Fitting

The Graduate Center, City University of New York, Depts. of Computer Science, Philosophy, Mathematics (emeritus), 365 Fifth Avenue, New York, NY 10016, United States

ARTICLE INFO

Article history:
Received 6 October 2015
Received in revised form 24 March 2016
Accepted 26 March 2016
Available online 6 April 2016

MSC:
03A99
03B45
03B53
03B60

Keywords:
Justification logic
Modal logic
Realization

ABSTRACT

Justification logics connect with modal logics, replacing unstructured modal operators with *justification terms* explicitly representing interdependence and flow of reasoning. The number of justification logics quickly grew from an initial single instance to a handful to about a dozen examples. In this paper we provide very general, though partly non-constructive, methods that cover all previous examples, and extend to an infinite family of modal logics. The full range of the phenomenon is not known. The extent to which constructive methods apply is also not known, but it is related to the availability of cut-free proof methods for modal logics.

1. Introduction

Justification logics are similar to modal logics, but with modal operators replaced by an infinite family of *justifications* that are intended to stand for explicit reasons. They are very fine-grained and, crucially, can internalize details of their own formal proofs. Justification logics are connected with modal logics via *realization* results. Roughly, realization replaces modal operators in a modal theorem with justification terms whose structure reflects the flow of reasoning involved in establishing the modal theorem. The first justification logic was LP, which was connected with the modal logic S4. This is discussed in the next section. The number of modal logics known to have justification counterparts with connecting realization theorems has grown over the years. We now see we are dealing with a general phenomenon whose extent is not known. In this paper we give a general approach to justification logics and to realization. In particular we show that the family of modal logics with justification counterparts is infinite.

Since justification logics are not as familiar as modal logics, we begin with a brief history. In particular we discuss how they came about in the first place—their original motivation. Then we give a very general

E-mail address: melvin.fitting@lehman.cuny.edu.

http://dx.doi.org/10.1016/j.apal.2016.03.005

approach to justification logics and realization. We note that there are now versions of quantified justification logics, but in this paper we confine our discussion entirely to propositional cases.

2. The origin story

The family of justification logics grew from a single instance, *the logic of proofs*, LP. This was introduced by Sergei Artemov as an essential part of his program to create an arithmetic provability semantics for intuitionistic logic. Features of LP have had a significant influence on research into the developing group of justification logics, so it is appropriate that we begin with a brief discussion of LP history.

Gödel formulated, at least implicitly, a program to find an arithmetic semantics for intuitionistic logic. In a well-known note, [19], he introduced the modern axiomatization of the modal logic S4, thinking of the $\Box$ operator as an informal provability operator. He also gave an embedding from intuitionistic logic to S4: put $\Box$ before every subformula. This amounts to thinking of intuitionistic truth as a kind of informal provability, with provability conditions reflected in conditions imposed on $\Box$. But he also noted that S4 does not embed into Peano arithmetic, translating $\Box$ as his formal provability operator. If it did, then the S4 theorem $\Box\bot \supset \bot$ would turn into a provable statement asserting consistency, something ruled out by his famous second incompleteness theorem. Since then it has been learned that the logic of formal arithmetic provability is GL, Gödel–Löb logic, but this does not relate in the desired way to intuitionistic logic.

In [20], Gödel introduced the idea that instead of thinking of the $\Box$ operator of S4 as *provability*, it could be thought of as an *explicit* proof representative. Each occurrence of $\Box$ could be translated in a different way. While $(\exists x)(x \mathsf{\ Proves\ } \bot) \supset \bot$ is not provable in Peano arithmetic, for each n that is the Gödel number of an arithmetic proof, $(n \mathsf{\ Proves\ } \bot) \supset \bot$ is provable. In a sense this moves the existential quantifier into the metalanguage. Using explicit proof representatives, an embedding of S4 into arithmetic should be possible, Gödel suggested. This proposal was not developed further by Gödel, and his observations were not published until many years later when his collected works appeared. By this time the idea had been rediscovered independently by Sergei Artemov. Artemov's formal treatment involved the introduction of a new logic, LP, standing for *logic of proofs*. This is a modal-like language, but with *proof terms* which one could think of as encapsulating explicit proofs. It was necessary to show that LP embedded into formal arithmetic, and this was done in Artemov's *Arithmetic Completeness Theorem* which we do not discuss here. But it was also necessary to show that S4 embedded into LP. This involved the formulation and proof of Artemov's *Realization Theorem*. A proper statement will be found in Section 4. The definitive presentation of all this is in [1].

The methods that connected S4 with LP could also make connections between the standard modal logics, K, K4, T, and some others and weaker versions of LP. The Artemov Realization Theorem extended to these logics as well, essentially by leaving cases out. There was also an arithmetic interpretation because these were sublogics of S4, but the connection with arithmetic was beginning to weaken. The term *justification logics* began being used because, while the connection with formal provability was fragmenting, proof terms (now called *justification terms*) still had the role of supplying explicit justifications for (epistemically) necessary statements. Two fairly comprehensive treatments of justification logics like these, and not just of the logic of proofs, can be found in [2] and [4].

The logic S5 extended the picture in a significant way. A justification logic counterpart was created in [26–28], and a realization theorem was proved. However, the resulting justification logic did not have a satisfactory arithmetical interpretation, and the proof of realization was not constructive. A non-constructive, semantic, proof of realization had been given in [9] for S4. It also applied to standard weaker logics without significant change. The extension to S5 required new ideas involving *strong* evidence functions. This will play a role here as well. The original Artemov proof of realization, connecting S4 and LP, was constructive. Indeed, as a key part of Artemov's program to provide an arithmetic semantics for intuitionistic logic, it was essential that it be constructive. S5 was the first example where a non-constructive proof was the initial