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The poset of all copies of the random graph has the 2-localization property

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ABSTRACT

Let G be a countable graph containing a copy of the countable universal and homogeneous graph, also known as the random graph. Let $\mathrm{Emb}(G)$ be the monoid of self-embeddings of G, $\mathbb{P}(G) = \{f[G] : f \in \mathrm{Emb}(G)\}$ the set of copies of G contained in G, and $\mathcal{I}_G$ the ideal of subsets of G which do not contain a copy of G. We show that the poset $\langle \mathbb{P}(G), \subset \rangle$, the algebra $P(G)/\mathcal{I}_G$, and the inverse of the right Green's pre-order $\langle \mathrm{Emb}(G), \preceq^R \rangle$ have the 2-localization property. The Boolean completions of these pre-orders are isomorphic and satisfy the following law: for each double sequence $[b_{nm} : \langle n, m \rangle \in \omega \times \omega]$ of elements of $\mathbb{B}$

$$\bigwedge_{n\in\omega} \bigvee_{m\in\omega} b_{nm} = \bigvee_{\mathcal{T} \in \mathrm{Bt}(^{<\omega}\omega)} \bigwedge_{n\in\omega} \bigvee_{\varphi \in \mathcal{T} \cap {}^{n+1}\omega} \bigwedge_{k\leq n} b_{k\varphi(k)},$$

where $\mathrm{Bt}(^{<\omega}\omega)$ denotes the set of all binary subtrees of the tree $^{<\omega}\omega$.

1. Introduction

Let $\mathbb{X}$ be a relational structure, $\mathrm{Emb}(\mathbb{X})$ the monoid of its self-embeddings and $\mathbb{P}(\mathbb{X}) = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$ the set of copies of $\mathbb{X}$ inside $\mathbb{X}$ (more precisely, $\mathbb{P}(\mathbb{X})$ is the set of domains of the substructures of $\mathbb{X}$ which are isomorphic to $\mathbb{X}$). Defining two structures $\mathbb{X}$ and $\mathbb{Y}$ to be similar if $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y})$ we obtain a classification of structures and, for example, the structure $\langle \omega, \emptyset \rangle$, the linear order $\langle \omega, < \rangle$ and the complete graph $\langle \omega, \omega^2 \setminus \Delta_\omega \rangle$ are similar binary structures in this sense.

A coarser classification of structures, related to the right Green's pre-order $\preceq^R$ on the monoid $\mathrm{Emb}(\mathbb{X})$ (defined by $f \preceq^R g$ iff $f \circ h = g$, for some $h \in \mathrm{Emb}(\mathbb{X})$) is obtained by demanding that the posets $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$

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and $\langle \mathbb{P}(\mathbb{Y}), \subset\rangle$ are isomorphic. (It is easy to check that the antisymmetric quotient of the inverse of the right Green's pre-order, $\operatorname{asq}\langle \operatorname{Emb}(\mathbb{X}), (\preceq^R)^{-1}\rangle$, is isomorphic to the poset $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$, see [10].)

Finally, defining two structures $\mathbb{X}$ and $\mathbb{Y}$ to be similar if the Boolean completions of their posets of copies, $\operatorname{ro}\operatorname{sq}\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ and $\operatorname{ro}\operatorname{sq}\langle \mathbb{P}(\mathbb{Y}), \subset\rangle$ are isomorphic, we obtain a classification which is coarser that the previous two. Since isomorphism of the Boolean completions of posets of copies is the same as their forcing equivalence [9], the last classification is, in fact, the classification of the posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset\rangle$ determined by their forcing-related properties (see [4–8] for some results concerning the classification of countable relational structures).

The structure considered in this paper is the countable universal homogeneous graph (also known as the random graph, Erdős–Rényi graph [2], Rado graph [14]).

Concerning the classification of countable ultrahomogeneous relational structures we first mention the following result from [11] related to the poset of copies of the rational line, $\mathbb{Q}$, and the corresponding quotient $P(\mathbb{Q})/\operatorname{Scatt}$, where Scatt denotes the ideal of scattered suborders of $\mathbb{Q}$: if $\mathbb{S}$ denotes the Sacks perfect set forcing and $\operatorname{sh}(\mathbb{S})$ the size of the continuum in the Sacks extension, then for each countable non-scattered linear order L and, in particular, for the rational line, the poset $\mathbb{P}(L)$ is forcing equivalent to the two-step iteration

$$\mathbb{S} * \pi$$

where $1_{\mathbb{S}} \Vdash$ "π is a σ-closed forcing". If the equality $\operatorname{sh}(\mathbb{S}) = \aleph_1$ (implied by CH) or PFA holds in the ground model, then the second iterand is forcing equivalent to the poset $(P(\omega)/\operatorname{Fin})^+$ of the Sacks extension. Consequently,

$$\operatorname{ro}\operatorname{sq}\operatorname{asq}\langle \operatorname{Emb}(\mathbb{Q}), (\preceq^R)^{-1}\rangle \cong \operatorname{ro}\operatorname{sq}\mathbb{P}(\mathbb{Q}) \cong \operatorname{ro}((P(\mathbb{Q})/\operatorname{Scatt})^+) \cong \operatorname{ro}(\mathbb{S} * \pi).$$

The following similar statement for countable non-scattered graphs (that is, the graphs containing a copy of the Rado graph) was obtained in [12].

Theorem 1.1. *For each countable non-scattered graph $\langle G, \sim\rangle$ and, in particular, for the Rado graph, the poset $\mathbb{P}(G)$ is forcing equivalent to the two-step iteration*

$$\mathbb{P} * \pi \tag{1}$$

where $1_{\mathbb{P}} \Vdash$ "π is an ω-distributive forcing" and the poset $\mathbb{P}$ adds a generic real, has the $\aleph_0$-covering property (thus preserves ω_1), has the Sacks property and does not produce splitting reals. In addition,

$$\operatorname{ro}\operatorname{sq}\operatorname{asq}\langle \operatorname{Emb}(G), (\preceq^R)^{-1}\rangle \cong \operatorname{ro}\operatorname{sq}\mathbb{P}(G) \cong \operatorname{ro}(P(R)/\mathcal{I}_R)^+ \cong \operatorname{ro}(\mathbb{P} * \pi) \tag{2}$$

and these complete Boolean algebras are weakly distributive.[1]

We note that the Sacks forcing has all the properties listed in Theorem 1.1: it adds a generic real, has the $\aleph_0$-covering and the Sacks property and does not produce splitting reals. In the present paper we show that the poset of copies of the Rado graph and, hence, the forcing $\mathbb{P}$ from (1) share one more property with the Sacks forcing – the 2-localization property, first introduced and studied by Newelski and Rosłanowski in

[1] A complete Boolean algebra $\mathbb{B}$ is called *weakly distributive* (or $(\omega, \cdot, {<}\omega)$-*distributive*, see [3]) iff for each cardinal κ and each matrix $[b_{n\alpha} : \langle n, \alpha\rangle \in \omega \times \kappa]$ of elements of $\mathbb{B}$ we have

$$\bigwedge_{n\in\omega} \bigvee_{\alpha\in\kappa} b_{n\alpha} = \bigvee_{s:\omega\to[\kappa]^{<\omega}} \bigwedge_{n\in\omega} \bigvee_{\alpha\in s(n)} b_{n\alpha}.$$