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Square and Delta reflection

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A R T I C L E I N F O

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Starting from infinitely many supercompact cardinals, we force a model of ZFC where $\aleph_{\omega^2+1}$ satisfies simultaneously a strong principle of reflection, called Δ-reflection, and a version of the square principle, denoted $\square(\aleph_{\omega^2+1})$. Thus we show that $\aleph_{\omega^2+1}$ can satisfy simultaneously a strong reflection principle and an anti-reflection principle.

1. Introduction

One of the most fruitful research areas in set theory is the study of the so-called *reflection principles*. A reflection principle is, roughly, a statement establishing that for a certain class of structures, if the structure satisfies a given property, then there is a substructure of smaller cardinality that satisfies the same property. Reflection principles can be seen as versions of Downward Lowenheim–Skolem theorem and they derive from large cardinal notions such as the notion of strongly compact cardinal. Magidor and Shelah introduced in 1994 (see [6]) a two-cardinals reflection principle, denoted $\Delta_{\lambda,\kappa}$ where $\lambda < \kappa$ (see the definition in Section 2). This is a strong reflection principle as it implies many interesting properties of structures of various kind. For instance, given a cardinal κ, if $\Delta_{\omega_1,\kappa}$ holds, then every almost free Abelian group of size κ is free; if $\Delta_{\lambda,\kappa}$ holds for every $\lambda < \kappa$ – we say that κ has the *Delta reflection* – then for every graph G of size κ, if every subgraph of smaller size has coloring number $\mu < \kappa$, then G itself has coloring number μ; similar properties hold for other kind of structures under the assumption that κ has the Delta reflection (see [9] and [6]). It should be pointed out that these statements are always true when κ is a singular cardinal (see [9]), thus we are interested in those *regular* cardinals that satisfy the Delta reflection.

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The Delta reflection implies another classical reflection principle, namely the *stationary set reflection* which for a cardinal κ established that for every stationary subset S of κ of points of cofinality less than λ, there exists $\alpha < \kappa$ such that $S \cap \alpha$ is stationary in α. Since the stationary set reflection fails at the successor of a regular cardinal, in particular the Delta reflection can only hold at successors of singular cardinals. The Delta reflection follows from weakly compact cardinals, however Magidor and Shelah showed in [6] that it is consistent that a small cardinal, namely $\aleph_{\omega^2+1}$, has the Delta reflection, assuming ZFC is consistent with the existence of infinitely many supercompact cardinals. Moreover, the results proven in [6], combined with other older results by Eklof [2], Milner and Shelah [7,9], imply that $\aleph_{\omega^2+1}$ is the smallest regular cardinal that can have the Delta reflection. Recent work by the first author and Magidor [3] shows that $\aleph_{\omega^2+1}$ can even satisfy simultaneously the Delta reflection and the tree property, which is another strong reflection principle (more precisely, assuming ZFC is consistent with the existence of infinitely many supercompact cardinals, then ZFC is consistent with $\aleph_{\omega^2+1}$ having both the Delta reflection and the tree property); on the other hand, it is shown in [3] that the Delta reflection at $\aleph_{\omega^2+1}$ does not imply the tree property at this cardinal (nor the tree property implies the Delta reflection), thus the Delta reflection and the tree property are independent.

In this paper we show that, assuming the consistency of infinitely many supercompact cardinals, it is possible to force a model of ZFC where $\aleph_{\omega^2+1}$ satisfies simultaneously the Delta reflection and a version of the square, denoted $\square(\aleph_{\omega^2+1})$ (see the definition in Section 2). Introduced by Todorčević in [12], $\square(\kappa)$ is an anti-reflection principle on a cardinal κ. For example, as demonstrated in [12], $\square(\kappa)$ implies the failure of the tree property at κ. It also implies the failure of the simultaneous reflection, namely every stationary set can be split into two disjoint stationary parts such that there is no ordinal $\alpha < \kappa$ in which they are both stationary (see Veličković [13]). Rinot [8] proved that an even stronger failure of the simultaneous reflection follows from $\square(\kappa)$, namely any stationary subset of κ can be partitioned into κ many pairwise disjoint stationary sets such that no two of them reflect simultaneously. It follows by a result of Solovay [11] that $\square(\lambda)$ fails if there is a λ-strongly compact cardinal, so in a way the existence of a $\square(\lambda)$ sequence bounds the amount of downward reflection that we can get on structures of size λ.

$\square(\kappa^+)$ is a consequence of the well-known *Jensen's square principle* $\square_\kappa$ (see Jensen [4]). The Delta reflection implies the failure of the weak square $\square^*_\kappa$ which is another weak consequence of $\square_\kappa$ (hence a fortiori the Delta reflection implies the failure of $\square_\kappa$), thus our result implies that one can have at $\aleph_{\omega^2+1}$ a good balance between a reflection principle and an anti-reflection principle.

2. Preliminaries and notation

In this section we give the definition of the Delta reflection and $\square(\lambda)$. Then we prove some preliminary results about the Delta reflection that will be used in the final proof of our main theorem.

Notation 2.1. Let $\kappa < \mu$ be two regular cardinals, we denote by $E^\mu_{<\kappa}$ the set $\{\alpha < \mu \mid \mathrm{cof}(\alpha) < \kappa\}$, and we denote by E^μ_κ the set $\{\alpha < \mu \mid \mathrm{cof}(\alpha) = \kappa\}$.

Notation 2.2. Let f be a function and A be a set, we denote by $f[A]$ the set $\{f(x) \mid x \in A \cap \mathrm{dom}\, f\}$.

Given a forcing notion $\mathbb{P}$ and two conditions $p, q \in \mathbb{P}$, we write $p \leq q$ when p is stronger than q. Given a cardinal κ, we recall that

- $\mathbb{P}$ is *κ-closed* if every decreasing sequence of less than κ many conditions in $\mathbb{P}$ has a lower bound;
- $\mathbb{P}$ is *κ-directed closed* if for every set of less than κ many pairwise compatible conditions in $\mathbb{P}$, it is possible to find a lower bound;
- $\mathbb{P}$ is *κ-distributive* if it does not add sequences of ordinals of length less than κ;