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Expansions of o-minimal structures by dense independent sets

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A B S T R A C T

Let $\mathfrak{M}$ be an o-minimal expansion of a densely ordered group and $\mathcal{H}$ be a pairwise disjoint collection of dense subsets of M such that $\bigcup\mathcal{H}$ is definably independent in $\mathfrak{M}$. We study the structure $(\mathfrak{M},(H)_{H\in\mathcal{H}})$. Positive results include that every open set definable in $(\mathfrak{M},(H)_{H\in\mathcal{H}})$ is definable in $\mathfrak{M}$, the structure induced in $(\mathfrak{M},(H)_{H\in\mathcal{H}})$ on any $H_0\in\mathcal{H}$ is as simple as possible (in a sense that is made precise), and the theory of $(\mathfrak{M},(H)_{H\in\mathcal{H}})$ eliminates imaginaries and is strongly dependent and axiomatized over the theory of $\mathfrak{M}$ in the most obvious way. Negative results include that $(\mathfrak{M},(H)_{H\in\mathcal{H}})$ does not have definable Skolem functions and is neither atomic nor satisfies the exchange property. We also characterize (model-theoretic) algebraic closure and thorn forking in such structures. Throughout, we compare and contrast our results with the theory of dense pairs of o-minimal structures.

0. Introduction

This paper continues to explore the central theme of our earlier works [9] and [10], whose origin lies in Miller and Speissegger [27], in which we examine extensions T' of well-behaved first-order theories T extending that of dense linear orders without endpoints (DLO)—typically, o-minimal T—in which good

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behavior is preserved. The properties that are investigated can be either topological or model theoretic. Every ordered structure comes equipped with the topology on definable sets induced by the order topology. Given this, following [27], it can be asked if the so-called open core of any model of the extended theory—that is, the structure with atomic formulas precisely for those definable open sets in the model of the extended theory—is o-minimal, and if so, whether the open core includes no additional open sets as the open core of the structure in the original language. Model-theoretic properties whose preservation (or lack thereof) are investigated include elimination of imaginaries, strong dependence, existence of atomic models and definable Skolem functions, and the exchange property (with respect to definable closure).

We now begin to make more precise the setting of this paper. Given a theory T extending DLO in a language $L \supseteq \{<\}$, we are interested in understanding (relative to T) extensions T' of T in languages extending L by unary relation symbols that are interpreted in models of T' as sets that are both dense and codense (that is, having empty interior) in the underlying sets of the models. The general goal is to understand the result of allowing various kinds of topological noise to be introduced into models of T. In order to avoid degeneracy, we want the base theory T to be sufficiently well behaved and rich. A natural case for first investigation is that T be complete, o-minimal and extend the theory of densely ordered groups. (Without group structure, differences in kinds of noise tend to disappear, and results tend to degenerate. To illustrate, the theory of the extension of DLO by any given pairwise disjoint family of dense-codense unary predicates is easily seen to be complete.)

There is a canonically "wild" example, namely, $T = \mathrm{Th}(\mathbb{R}, <, +, \cdot\,)$ and $T' = \mathrm{Th}(\mathbb{R}, <, +, \cdot, \mathbb{Q})$. The model theory of T is both well understood and quite well behaved; in particular, T is o-minimal, and so every open set (of any arity) definable in any model of T has only finitely many definably connected components. But $\mathbb{Z}$ is interdefinable with $\mathbb{Q}$ over $(\mathbb{R}, <, +, \cdot\,)$ (by Robinson [33]) and $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$ defines every real Borel set (see Kechris [20, 37.6]), in particular, every open subset of any finite cartesian power of $\mathbb{R}$ and every subset of any finite cartesian power of $\mathbb{Q}$. Thus, while the open sets definable in models of T are as simple as possible relative to the theory of ordered rings, the extension T' has a model where the definable open sets are as complicated as possible, as is the structure induced on the new predicate.

In contrast to the preceding example, if K is any proper real-closed subfield of $\mathbb{R}$ and $T' = \mathrm{Th}(\mathbb{R}, <, +, \cdot, K)$, then no model of T' defines any open set (of any arity) that is not definable in the reduct of the model to L (van den Dries [13, Theorem 5]); as in [9], we abbreviate this property by saying that T **is an open core of** T'. More generally, if $\mathfrak{M}$ is an o-minimal expansion of a densely ordered group, and A is dense in M and the underlying set of a proper elementary substructure of $\mathfrak{M}$, then $(\mathfrak{M}, A)$ is called a **dense pair** (of o-minimal expansions of densely ordered groups). By [13, Lemma 4.1], A is also codense in M. Dense pairs have been studied extensively (e.g., [13,9,27]) and $\mathrm{Th}(\mathfrak{M}, A)$ is well understood relative to T; in particular, if $\mathfrak{M}' \equiv \mathfrak{M}$ and $(\mathfrak{M}', A')$ is a dense pair, then $(\mathfrak{M}, A) \equiv (\mathfrak{M}', A')$ (by [13, 2.5]) and $\mathrm{Th}(\mathfrak{M})$ is an open core of $\mathrm{Th}(\mathfrak{M}, A)$ (see [9, Section 5]).

In this paper, we analyze an orthogonal complement (so to speak) to dense pairs, namely, expansions $(\mathfrak{M}, (H)_{H\in\mathcal{H}})$ of $\mathfrak{M}$ by dense subsets H of M such that $\mathcal{H}$ is pairwise disjoint and $\bigcup \mathcal{H}$ is (definably) independent over $\mathfrak{M}$. The canonical motivating example is $(\mathbb{R}, <, +, H)$, where H is a dense Hamel basis, that is, a dense subset of $\mathbb{R}$ that is a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Remarkably, the analysis of this rather special case is essentially as difficult as that of the general (thus explaining our use of "H" for dense independent sets).

Throughout: T denotes a complete o-minimal extension of the theory of densely ordered groups with a distinguished positive element in a language $L \supseteq \{<, +, -, 0, 1\}$. (The assumption of a distinguished positive element is primarily for later technical convenience and is not needed in order to state our main results.)

We let T^{pair} denote the theory of dense pairs of T (but note that T^{d} is used in [9,13]). Let $\mathcal{P}$ be a set of unary relation symbols P, none of which belong to L, and put $L_{\mathcal{P}} = L \cup \{\, P : P \in \mathcal{P}\,\}$. Let T^{indep} be the