Annals of Pure and Applied Logic 167 (2016) 707–724

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Proof theory for lattice-ordered groups

Nikolaos Galatos [a,1], George Metcalfe [b,*,2]

[a] Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208, USA
[b] Mathematical Institute, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

ARTICLE INFO

Article history:
Received 25 March 2014
Received in revised form 17 October 2015
Accepted 21 March 2016
Available online 20 April 2016

MSC:
06F15
06F20
03F05
08A50

Keywords:
Lattice-ordered groups
Proof theory
Hypersequent calculi
Cut elimination
Co-NP completeness

ABSTRACT

Proof-theoretic methods are developed and exploited to establish properties of the variety of lattice-ordered groups. In particular, a hypersequent calculus with a cut rule is used to provide an alternative syntactic proof of the generation of the variety by the lattice-ordered group of automorphisms of the real number chain. Completeness is also established for an analytic (cut-free) hypersequent calculus using cut elimination and it is proved that the equational theory of the variety is co-NP complete.

1. Introduction

A *lattice-ordered group* (*ℓ-group*) is an algebraic structure

$$(L, \wedge, \vee, \cdot, ^{-1}, 1)$$

such that $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, ^{-1}, 1)$ is a group, and $\cdot$ preserves the order in both arguments; i.e., $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ for all $a, b, c \in L$. It follows also from this definition that the lattice $(L, \wedge, \vee)$ is distributive and that $1 \leq a \vee a^{-1}$ for all $a \in L$. We refer to [1] for proofs and further standard facts about this class of structures.

* Corresponding author.
E-mail addresses: ngalatos@du.edu (N. Galatos), george.metcalfe@math.unibe.ch (G. Metcalfe).
[1] Supported by Simons Foundation grant 245805 and FWF project START Y544-N23.
[2] Supported by Swiss National Science Foundation (SNF) grant 200021_146748.

http://dx.doi.org/10.1016/j.apal.2016.04.004

Commutative ℓ-groups include the real, rational, and integer numbers with the standard total order and addition. For non-commutative examples, consider a chain (totally-ordered set) Ω and denote by $Aut(\Omega)$ the set of all order-preserving bijections on Ω. Then $Aut(\Omega)$ constitutes an ℓ-group $\mathbf{Aut}(\Omega)$ under coordinate-wise lattice operations, functional composition, and functional inverse. It was proved by Holland in [10] that the variety of ℓ-groups $\mathcal{LG}$ is generated by $\mathbf{Aut}(\mathbb{R})$, where $\mathbb{R}$ is the real number chain, or indeed by any $\mathbf{Aut}(\Omega)$, where Ω is an n-transitive chain for all n (i.e., for any two n-tuples of elements of Ω there is a bijection that maps the first tuple to the second). The standard proof relies on Holland's embedding theorem, which states, analogously to Cayley's theorem for groups, that every ℓ-group embeds into an ℓ-group $\mathbf{Aut}(\Omega)$ for some chain Ω [9]. Although not every ℓ-group embeds into $\mathbf{Aut}(\mathbb{R})$, each identity that fails in some ℓ-group fails, by the embedding theorem, in some automorphism ℓ-group, and a simple argument then shows that the identity must also fail in $\mathbf{Aut}(\mathbb{R})$. This generation result for $\mathcal{LG}$ was subsequently exploited by Holland and McCleary to provide an algorithm for checking if an identity is valid in all ℓ-groups [11].

The first main contribution of this paper is a new syntactic (and first axiom of choice free) proof that $\mathbf{Aut}(\mathbb{R})$ generates the variety $\mathcal{LG}$ of ℓ-groups. A proof system is defined in a one-sided hypersequent framework such that derivability of a hypersequent (interpreted as a disjunction of group terms) implies the validity of a corresponding identity in all ℓ-groups. A rule is then added to the system and it is shown, following closely the Holland–McCleary algorithm of [11], that this augmented system derives all identities (rewritten in a certain form) that are valid in $\mathbf{Aut}(\mathbb{R})$. Finally, it is proved syntactically that applications of this rule can be eliminated from derivations. Hence an identity is valid in $\mathbf{Aut}(\mathbb{R})$ if and only if it is valid in all ℓ-groups, and so, by Birkhoff's variety theorem, $\mathbf{Aut}(\mathbb{R})$ generates $\mathcal{LG}$. This proof illustrates the usefulness of proof-theoretic methods for tackling algebraic problems, and is similar to proofs of generation of varieties by dense chains via density elimination (see [4,14]) or of properties such as interpolation and amalgamation via cut elimination (see, e.g., [7,18]).

The second main contribution is the introduction of a first analytic (cut-free) proof calculus for ℓ-groups. In contrast to the well-developed proof theory for well-behaved families of varieties of residuated lattices (which provide algebraic semantics for substructural logics, see [2,3,7,14,17,18]), there has been relatively little success in obtaining cut-free systems for algebraic structures related to ℓ-groups. Hypersequent calculi have been defined for abelian ℓ-groups and related varieties in [13,15–17], but a calculus for the general non-commutative case has until now been lacking. The virtue of such a calculus is illustrated by the fact that we obtain not only the known decidability result for the equational theory of ℓ-groups, but also, via cut elimination, a (first) procedure for obtaining proofs of valid ℓ-group identities in equational logic (i.e., using only defining identities of $\mathcal{LG}$). More generally, the analytic hypersequent calculus presented here provides a crucial first step towards developing a uniform proof theory for the wide range of algebras and logics related in some way to ℓ-groups: in particular, MV-algebras and GMV-algebras (which may be viewed as intervals in abelian ℓ-groups [20] and ℓ-groups [6,8], respectively) and commutative cancellative residuated lattices (which may be viewed as ℓ-groups with a co-nucleus [19]).

The final contribution of the paper is a first proof that the equational theory of ℓ-groups is co-NP complete, matching the complexity of the equational theories of both abelian ℓ-groups [21] and distributive lattices [12].

2. Preliminaries

Let us call a variable x and its inverse x^{-1} *literals*. Using De Morgan identities valid in all ℓ-groups, we consider only *normalized* ℓ-group terms s, t built from literals and the operation symbols 1, $\wedge$, $\vee$, and $\cdot$, with an inductively defined inverse:

$$\begin{array}{rclcrcl} \overline{1} & = & 1 & \qquad & \overline{(s \cdot t)} & = & \overline{t} \cdot \overline{s} \\ \overline{x} & = & x^{-1} & & \overline{(s \wedge t)} & = & \overline{s} \vee \overline{t} \\ \overline{x^{-1}} & = & x & & \overline{(s \vee t)} & = & \overline{s} \wedge \overline{t}. \end{array}$$