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Quasiminimal structures, groups and Zariski-like geometries ☆,☆☆

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We generalize Hrushovski's Group Configuration Theorem to quasiminimal classes. As an application, we present Zariski-like structures, a generalization of Zariski geometries, and show that a group can be found there if the pregeometry obtained from the bounded closure operator is non-trivial.

1. Introduction

We will study quasiminimal classes, i.e. abstract elementary classes (AECs) that arise from a quasiminimal pregeometry structure. Abstract elementary classes were introduced to model theory by Shelah [26], and they provide the standard framework for the study of non-elementary classes. Quasiminimal classes originate from Zilber's [27] work, where he introduced quasiminimal excellent classes in order to prove categoricity of the non-elementary theory of pseudoexponential fields.

A quasiminimal class is an AEC that can be constructed from a infinite dimensional quasiminimal pregeometry structure (see [2] for details). Quasiminimal pregeometry structures provide an analogue to the strongly minimal first order setting. In the latter, a canonical pregeometry is obtained from the model theoretic algebraic closure operator, and ranks can be calculated as dimensions in the pregeometry. In the quasiminimal case, a pregeometry is obtained from the bounded closure operator (defined as Galois types

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having only boundedly many realizations, denoted bcl). A good exposition on quasiminimal pregeometries can be found in [23] and another one in [1].

Quasiminimal classes are uncountably categorical and have arbitrarily large models. They have both AP and JEP and thus a universal model homogeneous monster model. They are also excellent in the sense of Zilber (this is weaker than the original notion of excellence due to Shelah, see below).

Our main result, Theorem 3.10, is a generalization of Hrushovski's group configuration theorem to this setting. Hrushovski proved the group configuration theorem as a part of his Ph.D. thesis [5]. It has been the source of many applications of model theory to other fields of mathematics. The theorem holds for stable first order theories. It states that whenever there is a certain configuration of elements in a model, a group can be interpreted there. The proof can be found in e.g. [25]. In the present paper, we generalize it to the non-elementary setting of quasiminimal classes.

For the proof, we need an independence calculus that works in our context, so we develop it in Chapter 2. Independence in AECs has been previously studied under various assumptions. For instance, Hyttinen and Lessman [17] developed an independence notion for excellent (in the sense of Shelah) classes and showed that under the further assumption of simplicity, it satisfies the usual properties of non-forking. Hyttinen and Kesälä [14,15] developed an independence calculus under the related assumption that an AEC be finitary. There are also various other approaches to independence in AECs, some of them under very general assumptions (see e.g. [3]).

In the most general approaches, independence is considered mainly over models, usually rather saturated ones. However, we take a different direction and develop our independence notion with independence over finite tuples in mind. Indeed, this is what is needed for proving the group configuration theorem. For this purpose, we introduce FUR-classes ("Finite U-Rank"). The main reason for introducing these classes is to have a context in which certain variants of quasiminimal classes can be treated uniformly. We note that the previous approaches from [14] and [17] don't suffice for this, since all quasiminimal classes are not excellent in the sense of Shelah (as assumed in [17]) nor finitary in the sense of [14]. A counterexample for both criteria can be found in Example 2.3. In the elementary context, a model class of a first order ω-stable theory with finite Morley rank (with $\preccurlyeq$ the first order elementary submodel relation) provides an example of a FUR-class.

When developing the theory of independence, we use ideas from [17] and [18], and thus also from [10]. However, the classes treated there are excellent in the sense of Shelah, and quasiminimal classes don't satisfy all the hypothesis made there, so we cannot directly apply the results. The basic idea is that we first assume that we have a class with some kind of independence relation (given in terms of non-splitting) and then show that we actually also have another independence notion (given in terms of Lascar splitting, see Definition 2.38) that has all the properties we could expect from and independence notion (Theorem 2.70).

Hrushovski's group configuration theorem yields a group that consists of imaginaries (equivalence classes in definable equivalence relations), and canonical bases are used in the proof. Thus, in addition to having a theory of independence in the monster model $\mathbb{M}$, we need to construct $\mathbb{M}^{eq}$ and show that the theory can be applied there as well. We do this in section 2.4. In our context, $\mathbb{M}^{eq}$ cannot be constructed so that it is both ω-stable (in the sense of AECs) and has elimination of imaginaries. Since ω-stability is vital for our arguments, we build the theory (using ideas from [18]) so that we can always move from $\mathbb{M}$ to $\mathbb{M}^{eq}$ and then, if needed, to $(\mathbb{M}^{eq})^{eq}$ etc. We then show that if $\mathbb{M}$ is a monster model for a FUR-class, then so is $\mathbb{M}^{eq}$. At the end of section 2.4, we show that we have canonical bases and that they have the usual properties one would expect.

The main reason for presenting FUR-classes is to prove that quasiminimal classes have a perfect theory of independence, and in section 2.5 we show that every quasiminimal class is indeed a FUR-class. Moreover, we note that in quasiminimal classes, ranks can be calculated as dimensions in the pregeometry obtained from the bounded closure operator. However, we also give an example of a non-elementary class that is a FUR-class but does not arise from a quasiminimal pregeometry structure (Example 2.4).