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A problem of Laczkovich: How dense are set systems with no large independent sets?

Péter Komjáth [1]

Institute of Mathematics, Eötvös University, Budapest, P.O. Box 120, 1518, Hungary

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ABSTRACT

We investigate the function $L_{\mathcal{H}}(n) = \max\{|\mathcal{H} \cap \mathcal{P}(A)| : |A| = n\}$ where $\mathcal{H}$ is a set of finite subsets of λ such that every λ-sized subset of λ has arbitrarily large subsets form $\mathcal{H}$. For $\kappa = \aleph_1$, $\lim L_{\mathcal{H}}(n)/n \to \infty$ and $L_{\mathcal{H}}(n) = \mathrm{O}(n^2)$, and in different models of set theory, either bound can be sharp. If $\lambda > \aleph_1$, $L_{\mathcal{H}}(n) > cn^2$ for some $c > 0$ and n sufficiently large. If λ is strong limit singular, then $L_{\mathcal{H}}$ is superpolynomial. If $\kappa < \lambda$ are uncountable cardinals, we call a family $\mathcal{H}$ κ-dense (strongly κ-dense) in λ if every κ-sized subset of λ contains a set (arbitrarily large sets) in $\mathcal{H}$. We show under GCH that if $\mathcal{H}$ is κ^+-dense in κ^{+r} (r finite), then $L_{\mathcal{H}}(n)/n^r \to \infty$ ($\kappa = \omega$) and $L_{\mathcal{H}}(n) > cn^{r+1}$ ($\kappa > \omega$). We also give bounds for $L_{\mathcal{H}}(n)$ when $\mathcal{H}$ has large chromatic or coloring number.

Miklós Laczkovich raised the following question [7]. Assume that the system $\mathcal{H}$ consisting of finite subsets of ω_1 is *strongly dense*, i.e., for every uncountable subset X of ω_1, there are arbitrarily large members of $\mathcal{H}$ which are subsets of X. What is the behavior of the function

$$L_{\mathcal{H}}(n) = \max\left\{|\mathcal{H} \cap \mathcal{P}(x)| : x \in [\omega_1]^n\right\},$$

how fast must it converge to infinity? It is easy to see that $cn \leq L_{\mathcal{H}}(n) \leq 2^n$ $(n > n_0)$ for some $c > 0$. (The terms dense and strongly dense are ad hoc names used only in this paper in order to formulate the results.) The simplest way to construct a strongly dense system of finite sets of ω_1 is if we select a sequence $2 \leq r_0 < r_1 < \cdots$ of natural numbers and set $x \in \mathcal{H}$ iff $|x| = r_i$ for some $i < \omega$. In this case, however, $L_{\mathcal{H}}(n)$ is large for many values, namely, if $n = 2r_i$, then $L_{\mathcal{H}}(n)$ is at least

$$\binom{n}{\frac{n}{2}} \sim c\frac{1}{\sqrt{n}}2^n$$

E-mail address: kope@cs.elte.hu.

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where $c = \sqrt{2/\pi}$. The question is, therefore, if $L_{\mathcal{H}}$ can be uniformly small, e.g., polynomial, etc.

The function was first introduced (in a slightly different form) in the paper [1] of Erdős, Galvin, and Hajnal. They were only interested in uniform systems, specifically, what finite subsystems must occur in every uncountably chromatic system of 3-element sets. They also considered the following related question. If $\mathcal{H} \subseteq [\omega_1]^3$ is dense in ω_1, then what finite configurations must appear. Some of the problems raised in [1] have recently been solved in my paper [5].

In the first part of the paper we investigate Laczkovich's original question. We show that there is a strongly dense system $\mathcal{H}$ with $L_{\mathcal{H}}(n) = \mathrm{O}(n^2)$ (Theorem 1). For every strongly dense system $L_{\mathcal{H}}(n)/n \to \infty$ (Theorem 2), and conversely, if the Continuum Hypothesis holds, then for every monotonic $f : \omega \to \omega$ with $f(n) \to \infty$ there is a strongly dense system with the above condition such that $L_{\mathcal{H}}(n) = \mathrm{O}(nf(n))$ (Theorem 3). This statement is also consistent with $2^{\aleph_0} = \aleph_2$ (Theorems 5, 6). If, however, Martin's axiom MA_{ω_1} holds, then $L_{\mathcal{H}}(n) > cn^2$ $(n > n_0)$ holds for these systems (Theorem 4). In Lemma 7 we consider the weaker property that $\mathcal{H} \subseteq [\omega_1]^{<\omega}$ contains arbitrarily large subsets of every closed, unbounded subset of ω_1. We show that there is such an $\mathcal{H}$ with $L_{\mathcal{H}}(n) \le n - 1$ and $\limsup L_{\mathcal{H}}(n)/n \ge 1$ holds for every $\mathcal{H}$ with this property.

In the second part of the paper, we make remarks concerning the general case when the underlying set is some uncountable cardinal λ and $\mathcal{H}$, a system of finite subsets of λ is *dense* or *strongly dense in* λ, that is, every subset of λ of cardinality λ contains an element (arbitrarily large elements) of $\mathcal{H}$.

In Lemma 8 we show that every dense system is either uniform or strongly dense, at least, if we restrict to a subset of full cardinality. We then show that if $\lambda > \omega_1$, then $L_{\mathcal{H}}(n) > cn^2$ for some $c > 0$ and if $2^\kappa = \kappa^+$ then for every $\varepsilon > 0$ there exists a strongly dense $\mathcal{H}$ in κ^+ with $L_{\mathcal{H}}(n) < \varepsilon n^2$. If λ is a strong limit singular cardinal, $\mathcal{H} \subseteq [\lambda]^{<\omega}$ is strongly dense in λ, then $L_{\mathcal{H}}(n)$ increases faster than any polynomial of n.

In the next part of the paper we consider the following variant of the problem. Let $\kappa < \lambda$ be uncountable cardinals and let $\mathcal{H} \subseteq [\lambda]^{<\omega}$ be a set system such that for every $s \in [\lambda]^\kappa$, there is $x \in \mathcal{H}$ such that $x \subseteq s$ ($\mathcal{H}$ is κ-dense in λ) or $\sup\{|x| : x \in \mathcal{H}, x \subseteq s\} = \omega$ ($\mathcal{H}$ is strongly κ-dense in λ). We show that if GCH holds, $1 \le r < \omega$, $\mathcal{H}$ is κ^+-dense in κ^{+r}, and $|x| \ge r+1$ $(x \in \mathcal{H})$, then $L_{\mathcal{H}}(n)/n^r \to \infty$ $(\kappa = \omega)$ and $L_{\mathcal{H}}(n) > cn^{r+1}$ $(\kappa > \omega)$. Further, if $f(n) \to \infty$, then there is a cardinality and GCH preserving forcing extension in which there is an ω_1-dense system $\mathcal{H}$ in ω_2 with $|x| \ge 3$ $(x \in \mathcal{H})$. Finally, if $f(n) \to \infty$, there is a forcing extension with a strongly ω_1 dense $\mathcal{H}$ in ω_ω with $L_{\mathcal{H}}(n) = O\left(n^2 f(n)\right)$.

In the last part of the paper we consider the more general classes of set systems with large chromatic or coloring number. We show that if $\mathrm{Col}(\mathcal{H})$ is uncountable, then $\lim L_{\mathcal{H}}(n)/n = \infty$ (Theorem 17). This extends Theorem 2 as if $\mathcal{H} \subseteq [\omega_1]^{<\omega}$ is dense, then $\mathrm{Chr}(\mathcal{H}) > \omega$, consequently $\mathrm{Col}(\mathcal{H}) > \omega$. Theorem 18 is a result in the other direction: if $f(n) \to \infty$, then there is a system $\mathcal{H}$ of cardinality ω_1 with $\mathrm{Col}(\mathcal{H}) > \omega$ and $L_{\mathcal{H}}(n) = \mathrm{O}(nf(n))$, and also there is an system $\mathcal{H}$ of cardinality continuum with $\mathrm{Chr}(\mathcal{H}) > \omega$ and $L_{\mathcal{H}}(n) = \mathrm{O}(nf(n))$.

In Theorem 19 we show that if $\mathcal{H}$ is a system consisting of k-tuples with $\mathrm{Col}(\mathcal{H}) > \omega$, then $L_{\mathcal{H}}(n) > c_k n^{1+\frac{1}{k-1}}$ for n sufficiently large. Theorem 20 complements this result by showing that if κ is regular, $2 \le k < \omega$, then there is a hypergraph $\mathcal{H}$ consisting of k-tuples of κ^{+k-1}, with $\mathrm{Col}(\mathcal{H}) > \kappa$ and $L_{\mathcal{H}}(n) = \mathrm{O}(n^{1+\frac{1}{k-1}})$. A similar result for the chromatic number is proved in [1].

Notation. Definitions. If κ is an infinite cardinal, $\kappa = \aleph_\alpha$, then $\kappa^+ = \aleph_{\alpha+1}$ and $\kappa^{+r} = \aleph_{\alpha+r}$. If κ is a cardinal, $r < \omega$, we define $\exp_r(\kappa)$ by recursion on r as follows: $\exp_0(\kappa) = \kappa$, $\exp_{r+1}(\kappa) = 2^{\exp_r(\kappa)}$. If S is a set, κ a cardinal, $[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$, $[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$. If A, B are sets of ordinals, $A < B$ denotes that $\alpha < \beta$ holds for $\alpha \in A$, $\beta \in B$. If x is a finite set of ordinals, $|x| \ge 2$, then $e(x)$ is the set of the last two elements of x.

If A is a set of ordinals, then $\mathrm{Add}(\omega, A)$ is the forcing adding a Cohen real to each element of A, namely, $p \in \mathrm{Add}(\omega, A)$ iff p is a function, $\mathrm{Dom}(p) \in [A \times \omega]^{<\omega}$, $\mathrm{Ran}(p) \subseteq 2$, $q \le p$ iff q extends p.

A system $\mathcal{H}$ of sets is *uniform*, if $|x| = |y|$ holds for $x, y \in \mathcal{H}$.