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Turing meets Schanuel ☆

Angus Macintyre

QMUL, United Kingdom

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ABSTRACT

I show that all Zilber's countable strong exponential fields are computable exponential fields.

1. Introduction

Turing [52] applied Tarski's decidability of the theory of real closed fields to the decision problem for solvability of equations in Lie groups. I do not know if he knew Tarski's problem on the decidability of the real exponential, but his strengths in analysis [50] would certainly have equipped him to confront this problem, and indeed Zilber's more recent problems [60] on the complex exponential. It seems to me likely that he would have appreciated the work of the last 25 years on the logic of the real and complex exponentials [59,37,60]. In this paper I consider Schanuel's Conjecture, now fundamental to our understanding of the logic of the real and complex exponentials, from the standpoint of Turing computability. This reveals many challenging problems, some of which I solve. These will be revealed below. The following is one example.

Theorem. *Schanuel's Conjecture for the complex exponential is equivalent to the version in which one quantifies only over computable complex numbers, i.e. numbers of the form $a + ib$ where a, b are computable reals.*

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E-mail address: a.macintyre@qmul.ac.uk.

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This is related to the beautiful result of Kirby and Zilber [31] on the uniform Schanuel Conjecture for the real exponential field. Another main result of the present paper, given in the last section as Theorem 8.3, is:

Theorem. *Each of the countable strongly existentially closed exponential fields of Zilber is a computable exponential field.*

The proof of this is quite demanding, because of the paucity of modern foundational material on computable algebra and algebraic geometry. It is notable that no analogue of this theorem is known for models of the theory of the complex exponential field.

2. Preliminaries about the basic real structures

The "structures mères" are the real exponential field $\mathbb{R}_{\mathrm{exp}}$ and the complex exponential field $\mathbb{C}_{\mathrm{exp}}$. These are naturally construed as structures for the language of exponential rings, with the usual primitives $+, -, \cdot, 0, 1, E$. The basic equational axioms [54] are

(1) axioms for commutative unital rings;
(2) $E(0) = 1$;
(3) $E(x + y) = E(x)E(y)$

There is by now a sizable literature on these E-rings. For earlier work one should consult [54,36]. For a more recent treatment, one should consult the very useful papers [28,29] by Jonathan Kirby.

2.1. Decidability

The decidability of $\mathbb{R}_{\mathrm{exp}}$ has been a very influential problem in mathematical logic. Over the years, the structure of definable relations in $\mathbb{R}_{\mathrm{exp}}$ became the focal problem, and this led to the large-scale study of o-minimal theories [55], one of the outstanding successes of model theory.

What we currently know about $Th(\mathbb{R}_{\mathrm{exp}})$ is that it is unconditionally model-complete [59] and o-minimal, and both constructively model complete (i.e. there is an algorithm which given a formula produces an existential formula equivalent to it over the theory) and decidable if Schanuel's Conjecture for $\mathbb{R}$ is true [37]. We also know a relative of Schanuel's Conjecture [37] equivalent to the decidability of $Th(\mathbb{R}_{\mathrm{exp}})$. I do not expect to see an unconditional proof of the decidability of $Th(\mathbb{R}_{\mathrm{exp}})$. There are, however, unconditional upper bounds on the Turing degree of $Th(\mathbb{R}_{\mathrm{exp}})$ (see Corollary 2.1 below).

2.2. Restricted analytic functions

Understanding $Th(\mathbb{R}_{\mathrm{exp}})$ has been greatly facilitated by the study of related systems connected to subanalytic sets [55,22]. The simplest of these systems is the real field, with the following restricted exponential $E \restriction [0, 1]$ satisfying:

$E \restriction [0, 1](x) = exp(x)$ if $|x| \leq 1$
$E \restriction [0, 1](x) = 0$ if $|x| > 1$.

Let $\mathbb{R}_{\mathrm{exp}\restriction[0,1]}$ be the real field with this structure, in the language for ordered exponential rings. It is obvious that $\mathbb{R}_{\mathrm{exp}\restriction[0,1]}$ is definable in $\mathbb{R}_{\mathrm{exp}}$, and so o-minimal, and it is known [17] that $\mathbb{R}_{\mathrm{exp}}$ is <u>not</u> interpretable in $\mathbb{R}_{\mathrm{exp}\restriction[0,1]}$.

It will be convenient to consider each of the enrichments $\mathbb{R}_{\mathrm{exp}\restriction[0,n]}$ of $\mathbb{R}$, for $n \in \omega$, got by adjoining $E \restriction [0, n]$, which is exp on $[0, n]$ and 0 elsewhere on the positive real line. It is quite clear that each of these is bi-interpretable with $\mathbb{R}_{\mathrm{exp}\restriction[0,1]}$, and inherits o-minimality from $\mathbb{R}_{\mathrm{exp}}$. We have good reason also to