Annals of Pure and Applied Logic 167 (2016) 939–981

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Duality, non-standard elements, and dynamic properties of r.e. sets

V.Yu. Shavrukov

Nijenburg 24, 1081 GG Amsterdam, Netherlands

ARTICLE INFO

Article history:
Available online 20 May 2016

MSC:
primary 03D25
secondary 03C62, 03H15, 06D50

Keywords:
Recursively enumerable sets
The lattice of recursively enumerable sets
Duality for distributive lattices
Non-standard models of arithmetic
Index sets

ABSTRACT

We investigate the Priestley dual $(\mathscr{E}^*)^\star$ of the lattice $\mathscr{E}^*$ of r.e. sets modulo finite sets. Connections with non-standard elements of r.e. sets in models of 1st order true arithmetic as well as with dynamic properties of r.e. sets are pointed out. Illustrations include the Harrington–Soare dynamic characterization of small subsets, a model-theoretic characterization of promptly simple sets, and relations between the inclusion ordering of prime filters on $\mathscr{E}^*$ (a.k.a. points of $(\mathscr{E}^*)^\star$) and the complexity of their index sets.

0. Introduction

The lattice $\mathscr{E}^* = \mathscr{E}/\mathrm{fin}$ of r.e. sets modulo finite differences has seen much effort invested in its study over the last half-century or so. A variety of methods is employed in its investigation. The dual space of $\mathscr{E}^*$ made a brief appearance as an instrument in the proof of Theorem 4 in Alton [1]. To my best knowledge however this has thus far remained the only documented encounter between r.e. sets and duality for bounded distributive lattices. Adjacent branches of recursion theory, on the other hand, have experienced more systematic application of duality methods — see Nerode [31] and the more recent Selivanov [36]. The present paper aims to lend support to the view that the utility potential of duality in the study of r.e. sets is far from exhausted.

There is a certain kinship between $\mathscr{E}^*$ and the lattice Σ_1/T of Σ_1 sentences modulo provability in a consistent formal theory T such as Peano Arithmetic PA. This kinship manifests itself in the isomorphism between $\mathscr{E}$ and $\Sigma_1(\boldsymbol{x})/\mathrm{TA}$, the lattice of Σ_1 formulas with a single free variable modulo equivalence in $\mathbb{N}$, or, equivalently, modulo full True Arithmetic TA. In other words, r.e. sets are like $\Sigma_1(\boldsymbol{x})$ formulas. When you quotient $\mathscr{E}$ by finite differences, you are in fact strengthening TA by the requirement that $\boldsymbol{x}$ be non-standard,

E-mail address: v.yu.shavrukov@gmail.com.

http://dx.doi.org/10.1016/j.apal.2015.10.004
0168-0072/

for non-standard numbers are precisely those whose membership in an r.e. set is not affected by finite variations of the latter. Thus $\mathscr{E}^* \cong \Sigma_1(\boldsymbol{x})/(\mathrm{TA} + \boldsymbol{x} > \mathbb{N})$.

Our interest in the (Priestley) dual space $(\mathscr{E}^*)^\star$ of $\mathscr{E}^*$ was motivated by progress in Σ_1/T where the dual — or at least the underlying ordering of prime filters of Σ_1/T — is known as the *E-tree* and has been around for a long time — see Jensen & Ehrenfeucht [15, § 3] or Simmons [39]. Throughout the paper we point out similarities and differences between $(\mathscr{E}^*)^\star$ and E-trees of formal theories.

The typical prime filter of Σ_1/T is the collection of Σ_1 sentences holding in a given model of T. Similarly, a prime filter of $\mathscr{E}^*$ is the collection of $\Sigma_1(\boldsymbol{x})$ formulas that hold in a model of TA with a distinguished non-standard element $\boldsymbol{x}$. Thus models of $\mathrm{TA} + \boldsymbol{x} > \mathbb{N}$ play a role in the study of $(\mathscr{E}^*)^\star$ similar to the role of models of T for the E-tree.

While the subject matter of the present paper was inspired by developments in provability theory, none of the results herein have any immediate connection to provability. The choice of TA as the umbrella theory is to some extent arbitrary and reflects personal preferences. Another obvious candidate would be $\mathrm{TA}_2 = \mathrm{Th}_{\forall\exists}\,\mathbb{N}$ where certain model-theoretic issues are simpler than with full TA (see e.g. Hirschfeld [12] or Hirschfeld & Wheeler [14]), while $\Sigma_1(\boldsymbol{x})$ formulas behave exactly the same — at least as long as $\boldsymbol{x}$ does not vary. Schmerl & Shavrukov [35] show, among other things, some positive influence on investigations into models of TA_2 from connections we pursue in the present paper.

With the dual space on the one hand and non-standard elements on the other, many of our arguments exhibit a geometric/visual flavour — we include many illustrations to emphasize this point. Oftentimes this turns out to relate to what Harrington & Soare [9] call *dynamic properties* of r.e. sets (or arrangements thereof). These properties are hallmarked by being formulated in terms of speed at which elements enter an r.e. set, in other words, a key role is given to enumeration stages. Through relations such as 'at most total recursively later than', a model of TA allows for coarser measurement of when a generic element enters a given r.e. set than do the natural numbers. This allows some dynamic properties to find their model-theoretic equivalents, as well as gain alternative proofs of known equivalences to lattice-theoretic properties.

Our ambition in this enterprise is to suggest that bringing together r.e. sets, duality, and models of arithmetic can lead to some useful synergy. Keeping accessibility in mind, we spend somewhat more ink on the basics of these three ingredients than would be appropriate for readers well versed in respective fields.

We start section 1 with a brief introduction to Priestley duality for bounded (that is, possessing 0 and 1) distributive lattices. Next come some first basic properties of $(\mathscr{E}^*)^\star$. Most of the section is taken up by examples of dual-space characterizations of prominent classes of r.e. sets (or relations between those) such as r-maximal sets and major subsets.

Section 2 introduces models of TA into the fray. After recalling preliminary facts and definitions and articulating the connection between points of $(\mathscr{E}^*)^\star$ and models of $\mathrm{TA}+\boldsymbol{x} > \mathbb{N}$, we present a characterization of simple sets via end-extensions of models. This exploits earlier ideas of Hirschfeld, Wilkie, and Schmerl.

Section 3 turns to dynamic properties of r.e. sets using models of arithmetic as an instrument. With the help of an old lemma of Wilkie, we treat major and small subsets, re-obtaining the Harrington–Soare dynamic characterization of the latter. We also produce a model-theoretic equivalent to prompt simplicity. It turns out that the behaviour of promptly simple r.e. sets in models of TA is not unlike that of inconsistency statements in models of formal theories. Along the way we isolate the class of *hinged* prime filters which is also going to play a part in the succeeding section.

In section 4 we look at index sets of prime filters of $\mathscr{E}^*$ (= points of $(\mathscr{E}^*)^\star$). This section is directly inspired by the earlier ascent of the E-tree in Shavrukov & Solovay [37]. We establish the key Jump-the-Gap Lemma which relates the Turing complexity of hinged prime filters in an inclusion chain to the ordering of that chain, and draw some consequences for order types of branches through $(\mathscr{E}^*)^\star$.

Our line of approach to $\mathscr{E}^*$ should, I believe, at least be a plentiful source of new questions. Some evidence to that effect is presented in section 5.