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The members of thin and minimal Π^0_1 classes, their ranks and Turing degrees [☆]

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We study the relationship among members of Π^0_1 classes, thin Π^0_1 classes, their Cantor–Bendixson ranks and their Turing degrees; in particular, we show that any nonzero Δ^0_2 degree contains a member of rank α for any computable ordinal α. Furthermore we observe that the degrees containing members of thin Π^0_1 classes are not closed under join.

1. Introduction

A (computably bounded) Π^0_1 class can be thought of as the collection of paths $[T]$ through a computable binary tree T. Their interest stems from the fact that they code many constructions in mathematics. For example, given any Π^0_1 class $\mathcal{C}$, there is a computable real closed field whose collection of orderings is in 1–1 correspondence with the members of $\mathcal{C}$, in fact, in a many–one degree preserving way (see [12]). Another

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classic example is that the degrees of complete extensions of Peano Arithmetic are exactly those of certain kinds of Π^0_1 classes which separate an effectively inseparable pair of c.e. sets (see [15]). As a consequence the study of Π^0_1 classes is an important topic in mathematical logic, of relevance to proof theory, reverse mathematics, computable algebra and analysis, model theory, and algorithmic randomness. The survey paper [1] provides excellent introductions to theory and applications of Π^0_1 classes.

The present paper is part of a long term programme which seeks to understand the relationships between the degrees of members of Π^0_1 classes, the Cantor–Bendixson ranks of Π^0_1 classes, and the position of a Π^0_1 class in the lattice of Π^0_1 classes. For example, if a countable Π^0_1 class $\mathcal{C}$ has rank 1, then every member of the class is computable from $\mathbf{0}''$, and if $\mathcal{C} = [T]$ where T is a tree having no dead ends, the degrees of members of $\mathcal{C}$ are further constrained to be computable from the halting problem. This result was generalized to all computable ordinals by Cenzer, Clote, Smith, Soare and Wainer [3]. At the other extreme, if a nonempty Π^0_1 class $\mathcal{A}$ has no computable members, then for any $\boldsymbol{x} \geq \mathbf{0}'$, $\mathcal{A}$ has a member X with X' of degree $\boldsymbol{x}$, the case with $\boldsymbol{x} = \mathbf{0}'$ being known as the *Low Basis Theorem* (all in Jockusch and Soare [10]).

The particular question motivating this paper is the following:

Suppose that a degree $\boldsymbol{a}$ has a member of rank r. Does it also have members of other ranks? If so, then what can other ranks be?

This question was first tackled by Cenzer and Smith [2] who proved that if $\boldsymbol{a} \leq \mathbf{0}'$ then $\boldsymbol{a}$ has a member of rank 1. On the other hand, Downey [7] proved that there are degrees $\boldsymbol{b}$ with $\mathbf{0} < \boldsymbol{b} \leq \mathbf{0}''$ all of whose members have a fixed rank $r < \omega$ for any $r > 0$. In particular, there is such a $\boldsymbol{b}$ all of whose members have rank 1.

Cholak and Downey [5] generalized the Cenzer–Smith result. They showed that a computably enumerable and nonzero $\boldsymbol{a}$ contains a set X of rank α for any computable ordinal α. By rank α we mean that X belongs to a countable Π^0_1 class of Cantor–Bendixson rank α, *and* not to one of any rank $< \alpha$. Furthermore, $\boldsymbol{a}$ contains a point Y which is not in any countable Π^0_1 class.

Our first result is to extend the Cholak–Downey result to all Δ^0_2 degrees.

Theorem 1.1. *Suppose that $\mathbf{0} < \boldsymbol{a} \leq \mathbf{0}'$, and that α is a computable ordinal greater than zero. Then $\boldsymbol{a}$ contains a set X of rank α.*

Our remaining results concern what are called *thin* Π^0_1 classes. These classes are the analogues of maximal sets (or more properly hyperhypersimple sets) for the lattice of Π^0_1 classes. They are defined as infinite classes $\mathcal{C}$ such that for any Π^0_1 subclass $\mathcal{D} \subset \mathcal{C}$, there is a clopen set $\mathcal{F}$ with $\mathcal{D} = \mathcal{F} \cap \mathcal{C}$. From Cholak, Coles, Downey and Herrmann [6], we know that this is equivalent to saying that the lattice of Π^0_1 subclasses of $\mathcal{C}$ forms a boolean algebra. Under duality, thin classes were first constructed by Martin and Pour-El [11] who constructed an essentially undecidable theory of propositions all of whose extensions were principal. In fact, they constructed a perfect thin Π^0_1 class. Perfect here means that there are no isolated points in the class. In passing, we remark that in [3] it is shown that these perfect thin classes form an orbit in the automorphism group of the lattice of Π^0_1 classes, and allow us to show that the "array noncomputable degrees" are invariant for this lattice in the same way that the high degrees are invariant via the maximal sets in the lattice of computably enumerable sets.

Next in Cenzer, Downey, Jockusch and Shore [4], it was shown that it is possible to construct a *minimal* class $\mathcal{M}$; a thin class of rank 1 with a unique rank one point. An equivalent formulation of minimality is that $\mathcal{M}$ is infinite and every Π^0_1 subclass is either finite or co-finite. The Cenzer–Smith phenomenon is not true for members of thin classes, that is, Cenzer et al. [4] showed that not every computably enumerable degree has members of thin classes: there is a c.e. $\boldsymbol{a}$ such that if $X \in \boldsymbol{a}$, then X is not thin. They also showed that for any computable α there is a set X of rank α and which lies on a thin class of rank α.

One of the goals of this project was to examine the relationship between thinness, ranks, and degrees. Our initial guess was that ranks and thinness should behave in a complex way. However, to our surprise, in the Δ^0_2 case, we found that thinness has nothing to do with rank.