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Simple structures axiomatized by almost sure theories

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ABSTRACT

In this article we give a classification of the binary, simple, ω-categorical structures with SU-rank 1 and trivial algebraic closure. This is done both by showing that they satisfy certain extension properties, but also by noting that they may be approximated by the almost sure theory of some sets of finite structures equipped with a probability measure. This study give results about general almost sure theories, but also considers certain attributes which, if they are almost surely true, generate almost sure theories with very specific properties such as ω-stability or strong minimality.

1. Introduction

For each $n \in \mathbb{N}$, let $\mathbf{K}_n$ be a non-empty finite set of finite structures equipped with a probability measure μ_n and let $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$. For any property $\mathbf{P}$ (often a sentence in the language) we may extend the measure μ_n to associate a probability to $\mathbf{P}$ by defining

$$\mu_n(\mathbf{P}) = \mu_n\{\mathcal{M} \in \mathbf{K} : \mathcal{M} \text{ satisfies } \mathbf{P}\}.$$

A property $\mathbf{P}$ such that $\lim_{n\to\infty} \mu_n(\mathbf{P}) = 1$ is said to be an **almost sure** property of (also called 'almost surely true in') $\mathbf{K}$. The set $\mathbf{K}$ is said to have a 0–1 **law** if, for each sentence φ in the language, either φ or $\neg\varphi$ is almost sure in $\mathbf{K}$ i.e. each formula has asymptotic probability 0 or 1. The almost sure theory of $\mathbf{K}$, denoted $T_{\mathbf{K}}$, is the set of all almost sure sentences. Notice that $\mathbf{K}$ has a 0–1 law if and only if $T_{\mathbf{K}}$ is complete.

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A theory is called ω-**categorical** if it has a unique countable model up to isomorphism. The following fact leads us to see that one method to show that a set $\mathbf{K}$ has a 0–1 law is to prove that $T_{\mathbf{K}}$ is ω-categorical.

Fact 1.1. *Let T be a theory which is categorical in some infinite cardinality. Then T has no finite models if and only if T is complete.*

Many 0–1 laws [2,6,9,13,14] are proved in this way and additionally the corresponding almost sure theories are supersimple with SU-rank 1 and have trivial algebraic closure. In this article we ask ourselves what the reason is for this pattern and whether the supersimple ω-categorical theories with SU-rank 1 tend to be almost sure theories. In [3] the author together with Koponen studied sets of SU-rank 1 in homogeneous simple structures with a binary vocabulary. In this case, a strong connection was found to both trivial algebraic closure but also to random structures and almost sure theories. The present article will explore these implications further and prove the following theorem.

Theorem 1.2. *If T is ω-categorical, simple with SU-rank 1 and trivial algebraic closure over a binary relational vocabulary then there exists a set of finite structures $\mathbf{K} = (\mathbf{K}_n, \mu_n)_{n\in\mathbb{N}}$ with a probability measure μ_n such that $T_{\mathbf{K}} = T$.*

The key to ω-categorical almost sure theories is the notion of extension properties. These are first order sentences which state that if we have a tuple of a certain atomic diagram, possibly satisfying certain extra properties, then we may extend this into a larger tuple which also satisfies certain specific first order formulas. The connection to ω-categorical theories is very clear as these properties describe how we stepwise should build an isomorphism, and the method has been used before to prove many previous 0–1 laws, [6,7,9,14,15] among others. It is possible to make the extension properties very specific and in this way we will get a characterization of the simple ω-categorical structures with SU-rank 1 with trivial algebraic closure by stating how their extension properties should look like. Furthermore the way the extension properties are created implies that these structures are not only ω-categorical but also homogenizable i.e. we may add a relational symbol to an $\emptyset$-definable relation to make it a homogeneous structure.

Studies of general almost sure theories have been done before, the most common is the extension of the Erdős–Rényi random graph which have been studied by Baldwin [5] among others. Their analysis is though quite different from what we apply in this article, especially because the almost sure theories they work with need not to be ω-categorical, though stable, as Baldwin points out.

A definable pregeometry is an especially interesting part in almost sure theories and simple theories. The 0–1 laws mentioned before all have trivial algebraic closure in their almost sure theories. However the author together with Koponen constructed in [4] a 0–1 law for structures which almost surely define a vector space pregeometry such that the structures also are restricted by certain coloring axioms. The question arises, why so few non-trivial pregeometries (or algebraic closures) are found in almost sure theories, and we will in this article partially answer it by two different results. One answer is that if we have simple enough extension properties (the common method to show 0–1 laws) then we will almost surely have a trivial algebraic closure in the almost sure theory. The other answer is that if the sets of structures $\mathbf{K}_n$ are such that $|\mathcal{N}| = n$ for all $\mathcal{N} \in \mathbf{K}_n$ then vector space pregeometries (or affine or projective geometries) do almost surely not exist. The pregeometries of simple structures are often vector space like (or affine/projective), and thus if we want to create simple ω-categorical almost sure theories with nontrivial algebraic closure, we may conclude from results in this article that we will need classes of structures which grow non-linearly together with more interesting extension properties.

There are many different ways to construct infinite structures from finite structures or by probabilistic methods. Fraïssé [10] showed that having a set of finite structures satisfying certain properties, among them amalgamation, generates a unique infinite homogeneous structure. This infinite structure contain