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Henkin sentences and local reflection principles for Rosser provability ☆

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ABSTRACT

In this paper, we investigate Rosser-type Henkin sentences, namely, sentences asserting their own provability in the sense of Rosser, and local reflection principles based on Rosser provability predicates. First, we give a necessary and sufficient condition that a sentence is a Rosser-type Henkin sentence of some Rosser provability predicate, and prove that any negated Rosser sentence can be a Rosser-type Henkin sentence. Secondly, we prove the existence of a Rosser provability predicate whose Rosser-type Henkin sentences are all provable or refutable. Thirdly, we solve the question raised by Shavrukov, and give a Rosser provability predicate whose local reflection principle is strictly weaker than the usual one. At last, we investigate the hierarchy of partial local reflection principles based on Rosser provability predicates.

1. Introduction

In 1952, Henkin [18] asked the question whether each sentence asserting its own provability in a theory T is provable or not. A sentence φ satisfying $T \vdash \varphi \leftrightarrow \mathsf{Pr}_T(\ulcorner \varphi \urcorner)$ is called a Henkin sentence of T, where $\mathsf{Pr}_T(x)$ is a canonical formula weakly representing T-provability. In 1955, Löb [16] answered to this question by proving the following theorem: for any sentence φ, if $\mathsf{Pr}_T(\ulcorner \varphi \urcorner) \to \varphi$ is provable in T, then φ is also provable in T. Therefore each Henkin sentence of T is provable in T.

However, we cannot say definitely that we have a complete picture of the nature of Henkin sentences. Kreisel [10] pointed out that the situation of the provability of Henkin sentences of non-canonical provability

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formulas can vary (see also [9]). Kreisel's indication becomes clearer by considering the so-called Rosser provability predicate $\mathsf{Pr}^R_T(x)$ which says that x has a T-proof whose code is smaller than the code of any T-proof of the negation of x. The idea of using such a formula was proposed by Rosser [17] to improve Gödel's first incompleteness theorem. It is known that for every sentence φ refutable in T, $\neg\mathsf{Pr}^R_T(\ulcorner\varphi\urcorner)$ is provable in T. Then $\varphi \leftrightarrow \mathsf{Pr}^R_T(\ulcorner\varphi\urcorner)$ is provable in T, and thus φ is a Henkin sentence of $\mathsf{Pr}^R_T(x)$. Therefore every provable or refutable sentence is a Henkin sentence of the Rosser provability predicate. A natural question arises: Is there an independent Henkin sentence based on $\mathsf{Pr}^R_T(x)$? In other words, can we prove an analogue of Löb's theorem that if φ is a Henkin sentence of the Rosser provability predicate, then φ is either provable or refutable?[1] The answer might not be simple because it is known that rearrangement of the order of non-standard proofs gives various Rosser provability predicates, and several properties of Rosser provability predicates are dependent on the choice of Rosser predicates (see [1,7,13,19]).

The local reflection principle $\mathsf{Rfn}(T)$ for T is the set $\{\mathsf{Pr}_T(\ulcorner\varphi\urcorner) \to \varphi : \varphi \text{ is a sentence}\}$ which can be seen as a schema expressing the soundness of T. Since $\mathsf{Pr}_T(\ulcorner 0 = \overline{1}\urcorner) \to 0 = \overline{1}$ is not provable in T by Gödel's second incompleteness theorem, $T + \mathsf{Rfn}(T)$ is a proper extension of T if T is consistent. On the other hand, it is known that T proves the Rosser consistency of T, and thus Gödel's second incompleteness theorem does not hold for Rosser provability predicates. Hence aspects of local reflection principles based on Rosser provability predicates can differ from those of the usual local reflection principles.

Goryachev [6] investigated local reflection principles based on Rosser provability predicates. Let $\mathsf{Rfn}^R(T)$ be the set $\{\mathsf{Pr}^R_T(\ulcorner\varphi\urcorner) \to \varphi : \varphi \text{ is a sentence}\}$. It is easy to see that $T + \mathsf{Rfn}(T)$ always includes $T + \mathsf{Rfn}^R(T)$. Goryachev proved that there is a Rosser provability predicate such that the theories $T + \mathsf{Rfn}(T)$ and $T + \mathsf{Rfn}^R(T)$ are deductively equivalent. However he did not give any example of a Rosser provability predicate such that $T + \mathsf{Rfn}(T)$ and $T + \mathsf{Rfn}^R(T)$ are not equivalent.

In this paper, we investigate Henkin sentences and local reflection principles based on Rosser provability predicates. In Section 2, we introduce several preliminary notions and notations, and describe the background of this paper.

Sections 3 and 4 concern Henkin sentences based on Rosser provability predicates. In Section 3, we give a necessary and sufficient condition that a sentence is a Henkin sentence for some Rosser provability predicate. From this result, we show that any negated Rosser sentence can be a Henkin sentence based on some Rosser provability predicate. In Section 4, we prove the existence of a Rosser provability predicate whose Henkin sentences are all provable or refutable. Thus from the results in Sections 3 and 4, we obtain that whether a Rosser provability predicate has an independent Henkin sentence is dependent on the choice of that predicate.

Sections 5 and 6 are devoted to the investigation of local reflection principles based on Rosser provability predicates. In Section 5, we solve the question raised by Shavrukov [19] concerning a proof predicate in which the order of non-standard proofs of unprovable sentences is particularly inscrutable, and give a Rosser provability predicate whose local reflection principle is strictly weaker than the usual one. In Section 6, we investigate the hierarchy of partial local reflection principles based on Rosser provability predicates, and prove the so-called unboundedness theorem. Also we improve Goryachev's result, that is, we prove that there is a Rosser provability predicate such that at each level in the arithmetical hierarchy, its partial local reflection principle is equivalent to the usual one. At last, we prove that whether the Σ_1 reflection principle includes the Π_1 reflection principle is dependent on the choice of a Rosser provability predicate, in contrast to the fact that the usual Σ_1 reflection principle always contains the usual Π_1 reflection principle.

[1] Henkin sentences based on Rosser provability predicates are also discussed in Halbach and Visser's [8]. Also the question of the existence of an independent Rosser-type Henkin sentence is raised in their paper (Question 7.2).