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Tameness, uniqueness triples and amalgamation

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A B S T R A C T

We combine two approaches to the study of classification theory of AECs:

(1) that of Shelah: studying non-forking frames without assuming the amalgamation property but assuming the existence of uniqueness triples and
(2) that of Grossberg and VanDieren [6]: (studying non-splitting) assuming the amalgamation property and tameness.

In [7] we derive a good non-forking λ^+-frame from a semi-good non-forking λ-frame. But the classes K_{λ^+} and $\preceq \restriction K_{\lambda^+}$ are replaced: K_{λ^+} is restricted to the saturated models and the partial order $\preceq \restriction K_{\lambda^+}$ is restricted to the partial order $\preceq^{NF}_{\lambda^+}$.
Here, we avoid the restriction of the partial order $\preceq \restriction K_{\lambda^+}$, assuming that every saturated model (in λ^+ over λ) is an amalgamation base and (λ, λ^+)-tameness for non-forking types over saturated models (in addition to the hypotheses of [7]): Theorem 7.15 states that $M \preceq M^+$ if and only if $M \preceq^{NF}_{\lambda^+} M^+$, provided that M and M^+ are saturated models.
We present sufficient conditions for three good non-forking λ^+-frames: one relates to all the models of cardinality λ^+ and the two others relate to the saturated models only. By an 'unproven claim' of Shelah, if we can repeat this procedure ω times, namely, 'derive' good non-forking λ^{+n} frame for each $n < \omega$ then the categoricity conjecture holds.
In [14], Vasey applies Theorem 7.8, proving the categoricity conjecture under the above 'unproven claim' of Shelah.
In [10], we apply Theorem 7.15, proving the existence of primeness triples.

1. Introduction

The notion of a good non-forking λ-frame was introduced by Shelah [12, II]. It is an axiomatization of the non-forking relation in superstable first order theories. The goal of the study of good non-forking frames is to classify AECs. If the amalgamation property does not hold then the definition of a Galois-type is problematic. So Shelah added the amalgamation property to the axioms of a good non-forking frame.

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Shelah [12, II.3] found cases, where we can prove the amalgamation property in a specific cardinality λ and to prove the existence of a non-forking relation, relating to models of cardinality λ. This is the reason, why Shelah defined the non-forking relation in a good non-forking frame as relating to models of a specific cardinality, λ, only (so the amalgamation in λ property is one of the axioms of a good non-forking λ-frame, but the amalgamation property is not!).

Shelah [12, II] presented a way to extend a good non-forking λ-frame to models of cardinality greater than λ and proved that several axioms are preserved. But the amalgamation property and Axioms 1.1 are hard to be proved even for models of cardinality λ^+.

Axioms 1.1.

(1) Extension,
(2) Uniqueness,
(3) Basic stability and
(4) Symmetry.

We now consider models of cardinality λ^+ only. In order to get the amalgamation property and Axioms 1.1, there were introduced two approaches:

(1) Shelah's approach: to change the AEC, such that the amalgamation and Axioms 1.1 will be satisfied,
(2) the tameness approach for non-forking frames: to add the tameness property to the hypotheses.

In Shelah's approach, the relation $\preceq \restriction K_{\lambda^+}$ is restricted to the relation $\preceq^{NF}_{\lambda^+}$ (see Definition 6.9). One advantage of the relation $\preceq^{NF}_{\lambda^+}$ is that $(K_{\lambda^+}, \preceq^{NF}_{\lambda^+})$ satisfies the amalgamation property (even if $(K_{\lambda^+}, \preceq \restriction K_{\lambda^+})$ does not satisfy the amalgamation property). So we get artificially the amalgamation in λ^+ property. But a new problem arises: the pair $(K_{\lambda^+}, \preceq^{NF}_{\lambda^+})$ may not satisfy smoothness (one of the axioms of AEC). In order to solve this problem, the class of models of cardinality λ^+ is restricted to the saturated models of cardinality λ^+ over λ (and we assume that there are not many models of cardinality λ^{++}).

Shelah [12, II] derived a good non-forking λ^+-frame, using Shelah's approach: he proved that in the new AEC (the class of saturated models with the relation $\preceq^{NF}_{\lambda^+}$) all the axioms of a good non-forking λ^+-frame are satisfied, assuming additional hypotheses. Jarden and Shelah [7, Theorem 11.1.5] generalized the work done in [12, II]: they introduced the notion of a semi-good non-forking λ-frame. It is a generalization of a good non-forking λ-frame, where the stability hypothesis is weakened. Jarden and Shelah proved that we can derive a good non-forking λ^+-frame, from a semi-good non-forking λ-frame, assuming similar additional hypotheses.

In order to clarify the importance of Shelah's approach to the solution of the categoricity conjecture, we have to recall the following definition: Roughly, we say that a good non-forking λ frame is n-successful when we can derive a good non-forking λ^{+m}-frame for each $m \leq n$ (for a precise definition, see [7, Definition 10.1.1]). ω-successful means n-successful for every $n < \omega$.

Shelah [12, III.12.40] claims the following (he did not publish a proof yet):

Conjecture 1.2. *Assume that $2^\lambda < 2^{\lambda^+}$ for each cardinal λ. Let $(K, \preceq)$ be an AEC such that there is an ω-successful good non-forking λ-frame with underlying class K_λ. Then K is categorical in some $\mu > \lambda^{+\omega}$ if and only if K is categorical in each $\mu > \lambda^{+\omega}$.*

The main advantage of Shelah's approach is that we do not assume that the amalgamation property holds.

Shelah's approach has two disadvantages: