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Nonstandardness and the bounded functional interpretation

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ABSTRACT

The bounded functional interpretation of arithmetic in all finite types is able to interpret principles like weak König's lemma without the need of any form of bar recursion. This interpretation requires the use of *intensional* (rule-governed) majorizability relations. This is a somewhat unusual feature. The main purpose of this paper is to show that if the base domain of the natural numbers is extended with nonstandard elements, then the bounded functional interpretation can be seen as falling out from a functional interpretation of nonstandard number theory without intensional notions. The original bounded functional interpretation can be seen as the trace left behind by the new interpretation when one sees it restricted to the standard number theoretical setting.

We also answer an open question regarding the conservativity of the transfer principle *vis-à-vis* functional interpretations of nonstandard arithmetic.

1. Introduction

The bounded functional interpretation was introduced in [5]. In its workings and definition, it relies on a systematic use of the Howard/Bezem (strong) majorizability notion. A somewhat unusual feature is the presence of rule-governed (as opposed to axiomatic-governed) primitive relations: the so-called *intensional* majorizability relations. This permits to show that bounded domains (in the sense of being bounded with respect to the intensional majorizability notion) enjoy some "compactness" properties, the paradigmatic example being the bounded domain of the Cantor space (thereby obtaining weak König's lemma). Of course, the presence of rules in the deductive apparatus obfuscates a clear semantic picture. In this paper we show that if the number theory is allowed to have nonstandard elements, then we can define a new bounded functional interpretation, this time without intensional notions, and recover from it the original bounded

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functional interpretation. In a sense, the nonstandard numbers provide a kind of "compactification" of the natural numbers, making them behave like a bounded domain.

The starting point for this paper was the recent functional interpretations of nonstandard arithmetic due to Benno van den Berg, Eyvind Briseid and Pavol Safarik [15]. Even a cursory look at the paper shows many similarities between their interpretation and the bounded functional interpretation (this is particularly striking in the intuitionistic case). As they say in the paper, their interpretation is inspired (in the classical case) by the similarities between Shoenfield's functional interpretation [13] and the "reduction algorithm" of Edward Nelson for converting proofs in IST (Internal Set Theory) into proofs in ZFC (see [12]). Neither the interpretation of Berg et al. nor Nelson's "reduction algorithm" is based on majorizability considerations. They are rather based on finiteness considerations. The ultimate goal of Berg et al. is to extract computational information – in the form of appropriate term witnesses – from proofs in the nonstandard systems. Their goal is feasible, but it comes with some costs. For instance, the transfer principle – a cornerstone of Nelson's interpretation – has a trivial "reduction" in Nelson's setting but, as pointed by Berg et al., does not have a term witnessing functional. (They conjecture that the transfer principle is nevertheless conservative over the base standard setting, but we show in Appendix A of this paper that this is not the case.) Given that they use term witnessing functionals, in the case of Berg et al. the road is open for replacing finiteness by majorizability (because majorizability arguments rely upon an appropriate theorem concerning the majorizability of closed terms, as in [8]). But why try making this replacement? Apart from the main objective of this paper, viz. to show that the bounded functional interpretation can be recast (without intensional notions) by considering the wider nonstandard setting, mere finiteness conditions are not as surprising as using majorizability notions because the interpretations based on the latter are able to validate so-called *uniform boundedness principles* (introduced in [9] and conveniently discussed in [10]), of which weak König's lemma is a consequence.

For the sake of brevity, this paper only studies classical theories. In the next section, inspired by (but not following) Berg et al., we introduce the finite type system $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ of nonstandard arithmetic and describe some pertinent principles. In Section 3, we define the new majorizability interpretation and prove a corresponding soundness theorem. Section 4 discusses the sense in which the bounded functional interpretation of [5] can be recovered from the interpretation of the nonstandard system. We also include a small Appendix A where we discuss the transfer principle, both in the new interpretation of this paper and in the interpretation of Berg et al.

2. Basic framework

Let $\mathsf{E\text{-}PA}^{\omega}$ be the theory of extensional Peano arithmetic in all finite types. We follow the treatment of [10] where there is only an equality for the base type 0. Equality at other types is defined extensionally and a pertinent axiom of extensionality is uphold. The main purpose of this section is to introduce an extension $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ of $\mathsf{E\text{-}PA}^{\omega}$. The language of this extension extends the language of $\mathsf{E\text{-}PA}^{\omega}$ by having unary predicates st^{σ} for each finite type σ (the predicates for *standardness*). Note that the terms of both languages remain the same. Before we proceed, let us give a word of caution: our theory $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ below differs from the theory $\mathsf{E\text{-}PA}^{\omega *}_{\mathrm{st}}$ of Berg et al. not only by not having types for finite sequences but, more importantly, because it has different axioms concerning the new predicates st^{σ} (note the second standardness axiom below).

The axioms of $\mathsf{E\text{-}PA}^{\omega}_{\mathrm{st}}$ are those of $\mathsf{E\text{-}PA}^{\omega}$ together with the *standardness axioms* and the *external induction rule*. Let us introduce some notations and make some observations. First of all, since we are working in classical logic, we adopt as our logical primitives $\vee$ (disjunction), $\neg$ (negation) and the universal quantifiers $\forall x^{\sigma}$. The other logical connectives are understood as being defined in the usual manner. We also adopt the complete deduction system for classical logic exposed in [13]. The Howard/Bezem notion of strong majorizability (introduced in [8] and [2]) is defined by induction on the finite type: