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Models of intuitionistic set theory in subtoposes of nested realizability toposes

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In [8] Joyal and Moerdijk have shown that realizability toposes over partial combinatory algebras (pca) host classes of *small maps* giving rise to initial ZF-algebras providing models of intuitionistic Zermelo–Fraenkel set theory IZF. Here we show that this can be extended to a much wider class of realizability toposes as considered in [4] and [3].
For this purpose we first show this result for *nested realizability toposes* $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ induced by a pca $\mathcal{A}$ together with a sub-pca $\mathcal{A}_\#$ as considered implicitly in [4] and then show that it is preserved by restriction to subtoposes. This suffices since all toposes considered in [4] and [3] arise as subtoposes of some nested realizability toposes.

0. Introduction

Given a *partial combinatory algebra* (pca) $\mathcal{A}$ (see e.g. [13]) together with a subpca $\mathcal{A}_\#$ of $\mathcal{A}$ we will construct the *nested realizability topos* $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ as described in [4] (without giving it a proper name there). It is well known (from e.g. [13]) that $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ appears as the exact/regular completion of its subcategory $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ of assemblies. In [4] the authors considered two complementary subtoposes of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$, namely the *relative realizability topos* $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_\#)$ and the *modified relative realizability topos* $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_\#)$, respectively.

Within nested realizability toposes we will identify a class of *small maps* giving rise to a model of intuitionistic set theory IZF (see [5,10]) as described in [8]. For this purpose we first identify a class of

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display maps in $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$ which using a result of [1] gives rise to the desired class of small maps in the exact/regular completion $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ of $\mathbf{Asm}(\mathcal{A}, \mathcal{A}_\#)$.

For showing that the subtoposes $\mathbf{RT}_r(\mathcal{A}, \mathcal{A}_\#)$ and $\mathbf{RT}_m(\mathcal{A}, \mathcal{A}_\#)$ also give rise to models of IZF we will prove the following general result. If $\mathcal{E}$ is a topos with a class $\mathcal{S}$ of small maps and $\mathcal{F}$ is a subtopos of $\mathcal{E}$ then there is a class $\mathcal{S}_\mathcal{F}$ of small maps in $\mathcal{F}$ which is obtained by closing sheafifications of maps in $\mathcal{S}$ under quotients in $\mathcal{F}$.

As explained in subsections 1.2.2 and 1.2.3 below this covers also the modified realizability topos as studied in [12] and the more recent Herbrand topos of van den Berg [3].

1. Nested realizability toposes and some of their subtoposes

Given a pca $\mathcal{A}$ in an elementary topos $\mathscr{S}$ we may construct the realizability topos $\mathbf{RT}_\mathscr{S}(\mathcal{A})$ relative to $\mathcal{S}$ as described in [13]. If $\mathscr{S}$ is the Sierpiński topos $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$ then a "nested pca", i.e. a pca $\mathcal{A}$ together with a subpca $\mathcal{A}_\#$ gives rise to a pca internal to $\mathbf{Set}^{\mathbf{2}^{\mathrm{op}}}$ from which one may construct the "nested realizability topos" $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ as described in [4,13].[1] Within $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ there is a unique nontrivial subterminal object u giving rise to the *open* subtopos induced by the closure operator $u \to (-)$ and the complementary subtopos induced by the closure operator $u \vee (-)$ as described in [4].

Next we will give more elementary descriptions of $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ and the above mentioned subtoposes.

1.1. The nested realizability topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$

Let $\mathcal{A}$ be a pca whose partial application is denoted by juxtaposition and $\mathcal{A}_\#$ be a subpca of $\mathcal{A}$, i.e. $\mathcal{A}_\#$ is a subset of $\mathcal{A}$ closed under application and there are elements k and s of $\mathcal{A}_\sharp$ such that for all $x, y, z \in \mathcal{A}$ it holds that $kxy = x$, $sxyz \simeq xz(yz)$ and sxy is always defined. We write i for skk and $\bar{k}$ for ki which, obviously, satisfy the equations $ix = x$ and $\bar{k}xy = y$, respectively. We write p, p_0 and p_1 for elements of $\mathcal{A}$ such that px_0x_1 is always defined and $p_i(px_0x_1) = x_i$ for $i = 0, 1$. For every natural number n we write $\underline{n}$ for the corresponding numeral as defined in [13]. Notice that k, $\bar{k}$, p, p_0, p_1 and the numerals $\underline{n}$ are all elements of $\mathcal{A}_\#$.

Since subsets of $\mathcal{A}$ are the propositions of the realizability topos $\mathbf{RT}(\mathcal{A})$ it is useful to fix some notation for the propositional connectives

$$
\begin{aligned}
&A \to B = \{a \in \mathcal{A} \mid ax \in B \text{ for all } x \in A\} \\
&A \wedge B = \{pxy \mid x \in A, y \in B\} \\
&A \vee B = (\{k\} \wedge A) \cup \big(\{\bar{k}\} \wedge B\big)
\end{aligned}
$$

Propositions of the nested realizability topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ will be pairs $A = (A_p, A_a) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A}_\#)$ such that $A_a \subseteq A_p$ where we call A_p and A_a the set of *potential* and *actual* realizers, respectively. We write $\Sigma(\mathcal{A}, \mathcal{A}_\#)$ for the set of these propositions. The above notation for propositional connectives is adapted to the current class of propositions as follows

$$
\begin{aligned}
&A \to B = (A_p \to B_p, \mathcal{A}_\# \cap (A_p \to B_p) \cap (A_a \to B_a)) \\
&A \wedge B = (A_p \wedge B_p, A_a \wedge B_a) \\
&A \vee B = (A_p \vee B_p, A_a \vee B_a)
\end{aligned}
$$

For the realizability tripos $\mathscr{P}(\mathcal{A})$ induced by the pca $\mathcal{A}$ see [13]. The nested realizability tripos $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)$ over $\mathbf{Set}$ induced by the nested pca $\mathcal{A}_\# \subseteq \mathcal{A}$ is defined as follows. For a set I the fibre $\mathscr{P}(\mathcal{A}, \mathcal{A}_\#)(I)$ is given by the set $\Sigma(\mathcal{A}, \mathcal{A}_\#)^I$ preordered by the relation $\vdash_I$ defined as

[1] In [4] they do not give a name to this topos and, moreover, write $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$ for the relative realizability subtopos of the nested realizability topos.