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Some supplements to Feferman–Vaught related to the model theory of adeles

Jamshid Derakhshan [a,*], Angus Macintyre [b]

[a] University of Oxford, Mathematical Institute, 24–29 St Giles', Oxford OX1 3LB, UK
[b] Queen Mary, University of London, School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, UK

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A B S T R A C T

We give foundational results for the model theory of $\mathbb{A}_K^{fin}$, the ring of finite adeles over a number field, construed as a restricted product of local fields. In contrast to Weispfenning we work in the language of ring theory, and various sortings interpretable therein. In particular we give a systematic treatment of the product valuation and the valuation monoid. Deeper results are given for the adelic version of Krasner's hyperfields, relating them to the Basarab–Kuhlmann formalism.

1. Introduction

We have recently revisited Weispfenning's work [25] on the rings of adeles $\mathbb{A}_K$ over number fields K. That work in turn depends on the classic paper of Feferman and Vaught [13] on generalized products. Our objective is to obtain the most refined analysis possible of definable sets in $\mathbb{A}_K$ (paying special attention to uniformity in K). One intended application is to computation of measures and integrals over $\mathbb{A}_K$. A first paper [10] on this will soon be available. We think of our approach as rather more geometric, and less abstractly model theoretic, than the analysis in [25] and [13]. We prefer to work in the language of ring theory (or sometimes topological ring theory), without the Boolean or lattice-theoretic scaffolding from

* Corresponding author.
E-mail addresses: derakhsh@maths.ox.ac.uk (J. Derakhshan), angus@eecs.qmul.ac.uk (A. Macintyre).

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[25] and [13] (which has much more general applicability). We wish to stress that we add little to the foundations of the theory of generalized products. The treatment in [25] and [13] can hardly be improved. Rather, we work directly on the adeles $\mathbb{A}_K$ as a ring. However, we depend on various quantifier eliminations for completions K_v and some of these are in many-sorted languages appropriate to Henselian fields, so we need a version of [13] for many-sorted structures. Moreover $\mathbb{A}_K$ is a restricted product in the sense of [13] even in a language (like ring-theory) with function-symbols. Though it is implicit in [13] how to deal with function symbols and sorts, we prefer to prepare a paper providing foundations appropriate to the adelic setting. Further motivation is provided by the model theory of the product valuation on $\mathbb{A}_K$. The image is a submonoid of a lattice ordered group, and so not literally itself a restricted product. But some simple technical work allows us to find a restricted product interpretation of the image of the valuation. So it is convenient to provide some foundational discussion appropriate to this case.

Much more interesting is our adelic version of the Basarab–Kuhlmann structures [1,20] on local fields. We relate this to Krasner's hyperrings [18] associated with local fields, and provide a natural quantifier elimination for the adelic version.

Finally, we address some stability-theoretic concepts for adeles: we show that the local fields are stably embedded in the adeles, the value monoid is not stably interpreted, and that the adeles does not have the negation of the tree property of the second kind called NTP_2.

All readers of [13] know the importance of enrichments of atomic Boolean algebras. A specially important case is $(Powerset(I), Fin)$, where $Powerset(I)$ is the powerset of I and Fin is the ideal of finite sets. In [9] we showed that there is a good elimination theory for various refinements, e.g. by *Even*, where *Even* picks out the finite sets of even cardinality, or by predicates expressing congruence conditions on cardinality of finite sets. We hope that these refinements will find applications.

We are able to work internally, in the language of ring theory, because $\mathbb{A}_K$ has lots of idempotents. It is not a von Neumann regular ring, so we are appealing to more than is used in the observations by Kochen [17] and Serre [24, p. 389] that an ultraproduct of fields is canonically isomorphic to the product of the fields modulo a maximal ideal.

2. Generalities

The data for theorems of Feferman–Vaught type consists of:

(i) A (possibly many-sorted) first-order language L, which has the equality symbol $=$ of various sorts, and may have relation symbols and function-symbols. Convenient references for many-sorted logic are [19,12,23,21]. One convention from [23] that we choose not to follow is that the sorts be disjoint. This is an unnecessary restriction, especially when the only well-formed equality statements in our formalism demand that the terms involved be of the same sort.

(ii) $\mathcal{L}_0$, the usual language for Boolean algebra, with primitives $\{0, 1, \wedge, \vee, ^{-}\}$,

(iii) $\mathcal{L}$, *any* extension of $\mathcal{L}_0$,

(iv) I, an index set, with associated atomic Boolean algebra $Powerset(I)$ (the powerset of I, which will be denoted by $\mathbb{B}$),

$\mathbb{B}_{\mathcal{L}}$ will be *some* $\mathcal{L}$-structure on $\mathbb{B}$ where $\{0, 1, \wedge, \vee, ^{-}\}$ have their usual interpretations.

(v) A family $\{M_i : i \in I\}$ of L-structures with product $\Pi = \prod_{i \in I} M_i$.

One first forms, for each sort σ, the product

$$\prod_{i \in I} Sort_\sigma(M_i),$$

where $Sort_\sigma(M_i)$, qua set, is just the σ-sort of M_i. This product is the σ-sort of the product Π. The elements are just functions f_σ on I with